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DUAL SPACES WITH THE KREIN-MILMAN PROPERTY HAVE THE RADON-NIKODÝM PROPERTY

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ABSTRACT. The assertion in the title is proved.

A Banach space Z has the *Radon-Nikodým Property* (RNP) if every Z-valued measure μ, on a finite measure space (S, Σ, λ), which is λ-continuous and of bounded variation has a Bochner-integrable derivative $f: S \to Z$. (Thus $\mu(E) = \int_E f d\lambda$ for each E in Σ.) The space Z has the *Krein-Milman Property* (KMP) if every bounded, closed, convex subset of Z is the closed convex hull of its extreme points. J. Diestel [3] has posed the following question:

(Q1) *Is the RNP equivalent to the KMP?*

The implication RNP $\Rightarrow$ KMP has been proved by J. Lindenstrauss (whose proof appears in [9]) but it is still unknown whether the converse holds in general. In this paper, we use a result of C. Stegall [11] to prove the following

Theorem. *Every dual Banach space with the Krein-Milman Property has the Radon-Nikodým Property.*

We remark that the RNP for a Banach space Z has recently been shown to be equivalent to each of the following geometric conditions:

(i) Every closed, bounded, convex subset of Z is the closed convex hull of its strongly exposed points. (A point x_0 in a convex set A is *strongly exposed* if there is a bounded linear functional f such that $f(x_0) = \max f(A)$ and such that $\|x_n - x_0\| \to 0$ whenever each x_n is in A and $f(x_n) \to f(x_0)$.)

(ii) Every bounded subset of Z is dentable. (A set B is *dentable* if, given $\epsilon > 0$, there exists x in B with x not in $\overline{\text{co}}\, \{y \in B: \|x - y\| > \epsilon\}$.)

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These equivalences follows from the work of several mathematicians. That (i) implies (ii), and that (ii) implies the RNP, were proved by M. A. Rieffel [10], who introduced the notion of dentability. H. Maynard [8] proved that the RNP is equivalent to the property that every bounded subset is "s-dentable", a property formally weaker than (ii). W. J. Davis and R. R. Phelps [2] showed that it is in fact equivalent to (ii), while it was shown in [5] that Maynard's proof could be modified to prove directly the equivalence of (ii) and the RNP. Phelps [9] proved that (ii) implies (i).

In 1940, N. Dunford and B. J. Pettis [4] showed that a separable dual space Z^* must have the RNP. Both J. J. Uhl, Jr. [12] and Maynard [8] have noted that the RNP is separably determined (i.e. X has the RNP if and only if every separable subspace of X does), and Uhl has posed the following question.

(Q2) *If X has the RNP, need every separable subspace of X be isomorphic to a subspace of a separable dual?*

C. Bessaga and A. Pełczyński [1] showed that separable dual spaces also have the KMP and asked the same question with KMP replacing RNP. It is now known that an arbitrary dual space Z^* has the RNP if and only if every separable subspace of Z has a separable dual. (The sufficiency is due to Uhl [12], and the necessity to Stegall [11].) It follows easily that if X is a dual space, then (Q2) has an affirmative answer. (From this and the main result of this paper, the Bessaga-Pełczyński question also has a positive answer for dual spaces.) However both (Q1) and (Q2) are still open for general spaces. In fact the following related question is still unsettled even for dual spaces.

(Q3) *Is the KMP separably determined* (*i.e., if every separable subspace of X has the KMP, need X have it*)?

Finally, we remark that for Banach spaces with unconditional basis, (Q1), (Q2), and (Q3) all have positive answers. (To prove this, it is sufficient to show that if Z has an unconditional basis and if every separable subspace of Z has the KMP, then every separable subspace of Z is isomorphic to a subspace of a separable dual. Since every separable subspace of Z is contained in a separable subspace with an unconditional basis, we may assume that Z itself is separable. If Z has the KMP, then Z contains no isomorphic copy of c_0 (since c_0 lacks the KMP) and so, by results of R. C. James [6] and S. Karlin [7], the basis of Z is boundedly complete and hence Z is isomorphic to a separable dual.)

Proof of the Theorem. If Δ is the Cantor set, then by the *dyadic intervals* of Δ we mean the sequence $A_1 = \Delta$, $A_2 = \Delta \cap [0, 1/3]$, $A_3 = \Delta \cap [2/3, 1]$, $A_4 = \Delta \cap [0, 1/9]$, $A_5 = \Delta \cap [2/9, 1/3], \cdots$. Thus, $\{A_n\}_{n=1}^{\infty}$ is a sequence of clopen subsets of Δ indexed such that for each n, A_n is the disjoint union of A_{2n} and A_{2n+1}. Thus, for each $k = 0, 1, 2, \cdots$, Δ is the disjoint union of $\{A_m: 2^k \leq m < 2^{k+1}\}$. The result of Stegall which we need is the following.

Stegall's theorem [11, Theorem 1]. *Suppose X is a separable Banach space with X^* nonseparable, and let $\epsilon > 0$ be given. Then there exist:* (i) *a subset Δ of the unit sphere of X^*, Δ weak* homeomorphic to the Cantor set*; *and* (ii) *a sequence (x_n) in X, with $\|x_n\| < 1 + \epsilon$ for all n, such that*

$$\sum_{n=1}^{\infty} \|Tx_n - \chi_{A_n}\| < \epsilon,$$

where $T: X \to C(\Delta)$ is the evaluation operator (*i.e.*, $(Tx)x^* = x^*(x)$) *and the A_n's are the* (*homeomorphic images of*) *the dyadic intervals of the Cantor set.*

Now suppose the dual space Z^* lacks the RNP. By the result of Uhl mentioned above, Z has a separable subspace X with X^* nonseparable. Apply Stegall's theorem with $\epsilon = \frac{1}{2}$.

Let λ be the Radon measure on Δ such that $\lambda(A_n) = 2^{-k}$ if $2^k \leq n < 2^{k+1}$. For each n, define a Radon measure λ_n on Δ by

$$\lambda_n(E) = \lambda(E \cap A_n)/\lambda(A_n),$$

for all Baire sets E. The evaluation operator T may be regarded as mapping X into $L^{\infty}(\Delta, \lambda)$. Since the latter space is injective, T has an extension to a bounded operator (still denoted by T) from all of Z into $L^{\infty}(\Delta, \lambda)$. The λ_n's may be regarded as members of $L^{\infty}(\Delta, \lambda)^*$. For each n, let $x_n^* = T^*(\lambda_n) \in Z^*$. We define the following sets:

$$C = \text{w}^*\text{-co}\{\lambda_n\} \subseteq L^{\infty}(\Delta, \lambda)^*,$$

$$D = \text{w}^*\text{-co}\{x_n^*\} \subseteq Z^*,$$

$$K = \{z^* \in D: z^*(x_n) \to 0 \quad \text{as } n \to \infty\}.$$

(Here "w*-co" means "weak*-closed convex hull".)

Both C and D are weak* compact convex sets and $D = T^*(C)$. We shall prove K is nonvoid, bounded, norm-closed, and convex but has no extreme points. This will complete the proof. That K is bounded and convex is clear. That it is norm-closed follows easily from the boundedness of the sequence (x_n).

We show that K is nonvoid. We have, for any n and m,

$$|x_n^*(x_m)| = |\lambda_n(Tx_m)| \leq |\lambda_n(Tx_m - \chi_{A_m})| + |\lambda_n(\chi_{A_m})|$$
$$\leq \|Tx_m - \chi_{A_m}\| + \lambda(A_n \cap A_m)/\lambda(A_n).$$

The last quantity tends to 0 as m tends to ∞ and so $K \supseteq \{x_n^*\}$. It only remains to show that K has no extreme points. First we show that K is an extremal subset of D. To see this, note that

$$x_n^*(x_m) = \lambda_n(Tx_m - \chi_{A_m}) + \lambda_n(\chi_{A_m}) \geq -\|Tx_m - \chi_{A_m}\|, \quad \forall n, m.$$

Hence $z^*(x_m) \geq -\|Tx_m - \chi_{A_m}\|$, $\forall z^* \in D$, $\forall m$ and therefore

$$\liminf z^*(x_m) \geq 0, \quad \forall z^* \in D.$$

Now suppose z_1^*, z_2^* are in D with $z^* = \frac{1}{2}(z_1^* + z_2^*)$ in K. Then

$$\limsup z_1^*(x_m) \leq 2 \limsup z^*(x_m) - \liminf z_2^*(x_m) \leq 0,$$

and so z_1^*, and similarly z_2^*, is in K. Therefore K is extremal in D.

To complete the proof it now suffices to show that if z^* is an extreme point of D, then z^* is not in K. The set $C \cap (T^*)^{-1}(z^*)$ is extremal in C (and nonvoid since $T^*(C) = D$). Hence there exists an extreme point β of C such that $T^*(\beta) = z^*$. Now β is in the weak* closure W of $\{\lambda_n\}$. However, it is easily seen from the definition of λ_n that $\lambda_n = \frac{1}{2}(\lambda_{2n} + \lambda_{2n+1})$, $\forall n$. Hence $\beta \in W \backslash \{\lambda_n\}$. Since $\lambda_n(\chi_{A_m})$ is either 0 or 1 whenever $n \geq m$, it follows that $\beta(\chi_{A_m})$ is either 0 or 1 for every m. Since

$$1 = \beta(\chi_{A_1}) = \sum_{m=2^k}^{2^{k+1}-1} \beta(\chi_{A_m}), \quad \forall k,$$

$\beta(\chi_{A_m}) = 1$ for infinitely many m. But for any m with $\beta(\chi_{A_m}) = 1$, we have

$$z^*(x_m) = \beta(Tx_m) = \beta(\chi_{A_m}) + \beta(Tx_m - \chi_{A_m}) > 1 - \epsilon = \tfrac{1}{2}.$$

Thus z^* is not in K and the proof is complete.

Finally, note that if K above were always norm-separable, then we would have given an affirmative answer to question (Q3) for dual spaces.

However, K is not always norm-separable as the following example demonstrates. Let Z ($= X$) be the space of continuous real functions on the Cantor set. Then, in Stegall's theorem above, Δ may be taken to be the natural embedding of the Cantor set into Z^*, and x_n to be χ_{A_n} for each n. Then each x_n^* is simply λ_n (acting as a functional on Z) and the set D is the set of all Radon probability measures on Δ. It is then easy to see that K is the set of atomless measures in D. But this set is not norm-separable.

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