DUAL SPACES WITH THE KREIN-MILMAN PROPERTY
HAVE THE RADON-NIKODÝM PROPERTY

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ABSTRACT. The assertion in the title is proved.

A Banach space $Z$ has the Radon-Nikodým Property (RNP) if every $Z$-valued measure $\mu$, on a finite measure space $(S, \Sigma, \lambda)$, which is $\lambda$-continuous and of bounded variation has a Bochner-integrable derivative $f: S \to Z$. (Thus $\mu(E) = \int_E d\lambda$ for each $E$ in $\Sigma$.) The space $Z$ has the Krein-Milman Property (KMP) if every bounded, closed, convex subset of $Z$ is the closed convex hull of its extreme points. J. Diestel [3] has posed the following question:

(Q1) Is the RNP equivalent to the KMP?

The implication $\text{RNP} \implies \text{KMP}$ has been proved by J. Lindenstrauss (whose proof appears in [9]) but it is still unknown whether the converse holds in general. In this paper, we use a result of C. Stegall [11] to prove the following

Theorem. Every dual Banach space with the Krein-Milman Property has the Radon-Nikodým Property.

We remark that the RNP for a Banach space $Z$ has recently been shown to be equivalent to each of the following geometric conditions:

(i) Every closed, bounded, convex subset of $Z$ is the closed convex hull of its strongly exposed points. (A point $x_0$ in a convex set $A$ is strongly exposed if there is a bounded linear functional $f$ such that $f(x_0) = \max f(A)$ and such that $\|x_n - x_0\| \to 0$ whenever each $x_n$ is in $A$ and $f(x_n) \to f(x_0)$.)

(ii) Every bounded subset of $Z$ is dentable. (A set $B$ is dentable if, given $\epsilon > 0$, there exists $x$ in $B$ with $x$ not in $\overline{\text{co} \{y \in B : \|x - y\| > \epsilon\}}$.)

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These equivalences follows from the work of several mathematicians. That (i) implies (ii), and that (ii) implies the RNP, were proved by M. A. Rieffel [10], who introduced the notion of dentability. H. Maynard [8] proved that the RNP is equivalent to the property that every bounded subset is "s-dentable", a property formally weaker than (ii). W. J. Davis and R. R. Phelps [2] showed that it is in fact equivalent to (ii), while it was shown in [5] that Maynard's proof could be modified to prove directly the equivalence of (ii) and the RNP. Phelps [9] proved that (ii) implies (i).

In 1940, N. Dunford and B. J. Pettis [4] showed that a separable dual space $Z^*$ must have the RNP. Both J. J. Uhl, Jr. [12] and Maynard [8] have noted that the RNP is separably determined (i.e. $X$ has the RNP if and only if every separable subspace of $X$ does), and Uhl has posed the following question.

(Q2) If $X$ has the RNP, need every separable subspace of $X$ be isomorphic to a subspace of a separable dual?

C. Bessaga and A. Pełczyński [1] showed that separable dual spaces also have the KMP and asked the same question with KMP replacing RNP. It is now known that an arbitrary dual space $Z^*$ has the RNP if and only if every separable subspace of $Z$ has a separable dual. (The sufficiency is due to Uhl [12], and the necessity to Stegall [11].) It follows easily that if $X$ is a dual space, then (Q2) has an affirmative answer. (From this and the main result of this paper, the Bessaga-Pełczyński question also has a positive answer for dual spaces.) However both (Q1) and (Q2) are still open for general spaces. In fact the following related question is still unsettled even for dual spaces.

(Q3) Is the KMP separably determined (i.e., if every separable subspace of $X$ has the KMP, need $X$ have it)?

Finally, we remark that for Banach spaces with unconditional basis, (Q1), (Q2), and (Q3) all have positive answers. (To prove this, it is sufficient to show that if $Z$ has an unconditional basis and if every separable subspace of $Z$ has the KMP, then every separable subspace of $Z$ is isomorphic to a subspace of a separable dual. Since every separable subspace of $Z$ is contained in a separable subspace with an unconditional basis, we may assume that $Z$ itself is separable. If $Z$ has the KMP, then $Z$ contains no isomorphic copy of $c_0$ (since $c_0$ lacks the KMP) and so, by results of R. C. James [6] and S. Karlin [7], the basis of $Z$ is boundedly complete and hence $Z$ is isomorphic to a separable dual.)
Proof of the Theorem. If $\Delta$ is the Cantor set, then by the dyadic intervals of $\Delta$ we mean the sequence $A_1 = \Delta$, $A_2 = \Delta \cap [0, 1/3]$, $A_3 = \Delta \cap [2/3, 1]$, $A_4 = \Delta \cap [0, 1/9]$, $A_5 = \Delta \cap [2/9, 1/3]$, $\ldots$. Thus, $\{A_n\}_{n=1}^\infty$ is a sequence of clopen subsets of $\Delta$ indexed such that for each $n$, $A_n$ is the disjoint union of $A_{2n}$ and $A_{2n+1}$. Thus, for each $k = 0, 1, 2, \ldots$, $\Delta$ is the disjoint union of $\{A_m: 2^k \leq m < 2^{k+1}\}$. The result of Stegall which we need is the following.

**Stegall's theorem** [11, Theorem 1]. Suppose $X$ is a separable Banach space with $X^*$ nonseparable, and let $\epsilon > 0$ be given. Then there exist: (i) a subset $\Delta$ of the unit sphere of $X^*$, $\Delta$ weak$^*$ homeomorphic to the Cantor set; and (ii) a sequence $(x_n)$ in $X$, with $\|x_n\| < 1 + \epsilon$ for all $n$, such that

$$\sum_{n=1}^\infty \|Tx_n - x_{A_n}\| < \epsilon,$$

where $T: X \rightarrow C(\Delta)$ is the evaluation operator (i.e., $(Tx)^* = x^*(x)$) and the $A_n$'s are the (homeomorphic images of) the dyadic intervals of the Cantor set.

Now suppose the dual space $Z^*$ lacks the RNP. By the result of Uhl mentioned above, $Z$ has a separable subspace $X$ with $X^*$ nonseparable. Apply Stegall's theorem with $\epsilon = \frac{1}{2}$.

Let $\lambda$ be the Radon measure on $\Delta$ such that $\lambda(A_n) = 2^{-k}$ if $2^k \leq n < 2^{k+1}$. For each $n$, define a Radon measure $\lambda_n$ on $\Delta$ by

$$\lambda_n(E) = \frac{\lambda(E \cap A_n)}{\lambda(A_n)},$$

for all Baire sets $E$. The evaluation operator $T$ may be regarded as mapping $X$ into $L^\infty(\Delta, \lambda)$. Since the latter space is injective, $T$ has an extension to a bounded operator (still denoted by $T$) from all of $Z$ into $L^\infty(\Delta, \lambda)$. The $\lambda_n$'s may be regarded as members of $L^\infty(\Delta, \lambda)^*$. For each $n$, let $x_n^* = T^*(\lambda_n) \in Z^*$. We define the following sets:

$$C = w^*\text{-co}\{\lambda_n\} \subseteq L^\infty(\Delta, \lambda)^*,$$

$$D = w^*\text{-co}\{x_n^*\} \subseteq Z^*,$$

$$K = \{z^* \in D: z^*(x_n) \rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

(Here "$w^*\text{-co}$" means "weak$^*$-closed convex hull").

Both $C$ and $D$ are weak$^*$ compact convex sets and $D = T^*(C)$. We shall prove $K$ is nonvoid, bounded, norm-closed, and convex but has no extreme points. This will complete the proof. That $K$ is bounded and convex
is clear. That it is norm-closed follows easily from the boundedness of the sequence \((x_n)\).

We show that \(K\) is nonvoid. We have, for any \(n\) and \(m\),
\[
|x_n^*(x_m)| = |\lambda_n(Tx_m)| \leq |\lambda_n(Tx_m - x_A_m)| + |\lambda_n(x_A_m)|
\]
\[
\leq \|Tx_m - x_A_m\| + \lambda(A_n \cap A_m) / \lambda(A_n).
\]
The last quantity tends to 0 as \(m\) tends to \(\infty\) and so \(K \supseteq \{x_n^*\}\). It only remains to show that \(K\) has no extreme points. First we show that \(K\) is an extremal subset of \(D\). To see this, note that
\[
x^*_n(x_m) = \lambda_n(Tx_m - x_A_m) + \lambda_n(x_A_m) \geq - \|Tx_m - x_A_m\|, \quad \forall n, m.
\]
Hence \(z^*(x_m) \geq - \|Tx_m - x_A_m\|, \forall z^* \in D, \forall m\) and therefore
\[
\liminf z^*(x_m) > 0, \quad \forall z^* \in D.
\]
Now suppose \(z_1^*, z_2^*\) are in \(D\) with \(z^* = \frac{1}{2}(z_1^* + z_2^*)\) in \(K\). Then
\[
\limsup z_1^*(x_m) \leq 2 \limsup z^*(x_m) - \liminf z_2^*(x_m) \leq 0,
\]
and so \(z_1^*\) and similarly \(z_2^*\), is in \(K\). Therefore \(K\) is extremal in \(D\).

To complete the proof it now suffices to show that if \(z^*\) is an extreme point of \(D\), then \(z^*\) is not in \(K\). The set \(C \cap (T^*)^{-1}(z^*)\) is extremal in \(C\) (and nonvoid since \(T^*(C) = D\)). Hence there exists an extreme point \(\beta\) of \(C\) such that \(T^*(\beta) = z^*\). Now \(\beta\) is in the weak* closure \(\overline{W}\) of \(\lambda_n\). However, it is easily seen from the definition of \(\lambda_n\) that \(\lambda_n = \frac{1}{2}(\lambda_{2n} + \lambda_{2n+1})\), \(\forall n\). Hence \(\beta \in \overline{W} \backslash \lambda_n\). Since \(\lambda_n(x_{A_m})\) is either 0 or 1 whenever \(n \geq m\), it follows that \(\beta(x_{A_m})\) is either 0 or 1 for every \(m\). Since
\[
1 = \beta(x_{A_1}) = \sum_{m=2^k}^{2^{k+1} - 1} \beta(x_{A_m}), \quad \forall k,
\]
\(\beta(x_{A_m}) = 1\) for infinitely many \(m\). But for any \(m\) with \(\beta(x_{A_m}) = 1\), we have
\[
z^*(x_m) = \beta(Tx_m) = \beta(x_{A_m}) + \beta(Tx_m - x_{A_m}) > 1 - \epsilon = \frac{1}{2}.
\]
Thus \(z^*\) is not in \(K\) and the proof is complete.

Finally, note that if \(K\) above were always norm-separable, then we would have given an affirmative answer to question (Q3) for dual spaces.
However, $K$ is not always norm-separable as the following example demonstrates. Let $Z ( = X)$ be the space of continuous real functions on the Cantor set. Then, in Stegall's theorem above, $\Delta$ may be taken to be the natural embedding of the Cantor set into $Z^*$, and $x^*_n$ to be $\chi_{A^n}$ for each $n$. Then each $x^*_n$ is simply $\lambda_n$ (acting as a functional on $Z$) and the set $D$ is the set of all Radon probability measures on $\Delta$. It is then easy to see that $K$ is the set of atomless measures in $D$. But this set is not norm-separable.

REFERENCES