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THE RELATIVE COMPLETION OF AN A-SEGAL ALGEBRA IS CLOSED

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ABSTRACT. The main result is this theorem: If the Banach algebra A has bounded approximate right units and B is an A-Segal algebra, then the relative completion of B with respect to A is an A-Segal algebra. Furthermore, B is a closed ideal of its relative completion with respect to A.

In this paper we define the relative completion of an A-Segal algebra and present an analysis of the resulting object. At the heart of our analysis is this fundamental result: In the presence of approximate units an A-Segal algebra is a closed ideal in its relative completion and the relative completion is itself an A-Segal algebra. This and the other results presented in this paper were announced in [4]. Furthermore, we will use the results in this paper in our analysis (done jointly with Richard R. Goldberg) of the multiplier theory of Segal algebras on locally compact groups [5].

It is of interest to compare the results in this paper with work of N. Th. Varopoulos [16].

Conventions. Algebras have no identity, and *ideal* means *left ideal* unless something to the contrary is explicitly stated. The reader will easily see how to formulate and prove "right-hand" versions of our theorems.

We begin our work with a statement of the definition of A-Segal algebra.

Definition 1. Let $(A, \| \|_A)$ be a Banach algebra. We say the subalgebra $B \subseteq A$ is an A-Segal algebra provided (i) B is a dense ideal of A which is a Banach algebra with respect to the norm $\| \|_B$, and (ii) the natural injection of B into A is continuous and multiplication is (jointly) continuous from $A \times B$ into B.

Remarks 2. Part (ii) of 1 implies the existence of constants $K, M > 0$ so that $\| \|_A \leq K\| \|_B$, $\|f \cdot g\|_B \leq M\|f\|_A\|g\|_B$ for all $f \in A$ and all $g \in B$. Although these latter inequalities are our basic tools, the phrasing in Def-

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inition 1 makes sense for topological algebras that are not necessarily normed algebras. The basic theory for the more general situation is indicated in [3] and many of the results herein are valid in the more general setting. Appealing to a result of M. A. Rieffel [14] we may and do take the constant M, appearing in the product inequality, to be one. Finally, as the results of the present paper constitute contributions to the structural aspects of A-Segal algebras, we point out that a great deal is already known about the structure of A-Segal algebras: [1]–[4], [6]–[9], [11], [13], [15].

Definition 3. Let B be an A-Segal algebra. The relative completion of B with respect to A, denoted $\tilde{B}^A$, is defined in this way:

$$\tilde{B}^A = \bigcup_{\eta>0} \overline{S_B(\eta)}^A,$$

where $S_B(\eta) = \{f \in B \mid \|f\|_B \leq \eta\}$ and $\overline{E}^A$ is the closure of E in the A-norm.

A norm on $\tilde{B}^A$, denoted $\||\ \||$, is defined in this way:

$$\||f\|| = \inf\{t \mid f \in \overline{S_B(t)}^A\}.$$

The following lemma provides us with a useful tool to be used in our analysis of $\tilde{B}^A$.

Lemma 4. *Let B be an A-Segal algebra. If $f \in \tilde{B}^A$, then there exists a sequence $\langle f_n \rangle$ in B such that*

(i) $\|f_n - f\|_A \to 0$, *and*

(ii) $\|f_n\|_B \to \||f\||$.

Proof. If $f \in \tilde{B}^A$ and $\alpha_k \equiv \||f\|| + 1/k$, $k = 1, 2, \cdots$, then $f \in \overline{S_B(\alpha_k)}^A$. Hence, there exists $f_n \in S_B(\alpha_n)$ so that $\|f_n - f\|_A < 1/n$, $n = 1, \cdots$, and so (i) follows. Now $\|f_n\|_B \leq \alpha_n$ and so we see that

(iii) $\text{Lim Sup}_{n\to\infty}\|f_n\|_S \leq \||f\||$.

Suppose, for some δ that $\text{Lim inf}_{n\to\infty}\|f_n\|_B < \delta < \||f\||$. Then some subsequence of $\langle f_n \rangle$ lies in $S_B(\delta)$. This and (i) would imply that $f \in \overline{S_B(\delta)}^A$, contradicting $\delta < \||f\||$. Hence, no such δ exists and so

$$\operatorname*{Lim\,inf}_{\eta\to\infty} \|f_n\|_B = \||f\||.$$

This and (iii) show that (ii) holds.

Here is an example of an A-Segal algebra in its relative completion. We set $A = L^1(G)$ and $B = C(G)$ where G is an infinite compact group. In this case $\widetilde{C(G)}^{L^1} = L^\infty(G) \neq C(G)$. We shall give further examples after some basic results have been established.

Now it is well known that the relative completion of a Banach space is again a Banach space [10]. In our situation more is true. Indeed, $\widetilde{B}^A$ is an A-Segal algebra. N. Th. Varopoulos [16] has shown that B need not be closed in $\widetilde{B}^A$. However, if B and A share right approximate units which have A norm one then B is closed in $\widetilde{B}^A$ and the embedding is an isometry. Furthermore, in the presence of right approximate units $\widetilde{B}^A$ has a simple description which is of interest in its own right (compare Theorem 5 below with [12, 35.11, p. 380]).

Convention. For the remainder of this paper we suppose that B and A have common right approximate units, E, with $\|e\|_A = 1$ for all $e \in E$.

Furthermore, we suppose that

$$\|fe\|_B \leq \|f\|_B \|e\|_A = \|f\|_B \qquad (f \in B,\ e \in E).$$

Note that this inequality is not a consequence of the definition of A-Segal algebra because B need not be a right ideal! This condition is satisfied in all the examples we know of.

Theorem 5. *If B is an A-Segal algebra, then the following two conditions are equivalent:*

(1) $f \in \widetilde{B}^A$;

(2) $M = \operatorname{Sup}\{\|fe\|_B \mid e \in E\} < \infty$.

If either condition holds then $M = |||f|||$.

Proof. Suppose $f \in \widetilde{B}^A$. By Lemma 4 there exists a sequence $\langle f_n \rangle$ in B so that $\|f_n - f\|_A \to 0$, and $\|f_n\|_B \to |||f|||$. Then, for each $e \in E$ we have $\|f_n e - fe\|_B \leq \|f_n - f\|_A \|e\|_B \to 0$ as $n \to \infty$. Thus, $\|fe\|_B = \operatorname{Lim}_{n\to\infty} \|f_n e\|_B \leq \operatorname{Lim}_{n\to\infty} \|f_n\|_B = |||f|||$. Hence, (2) holds and $M \leq |||f|||$. Now suppose (2) holds for some $f \in A$. Choose an (f-dependent) sequence $\langle e_n \rangle$ from E so that $\|fe_n - f\|_A \to 0$. By hypothesis $fe_n \in S_B(M)$. Hence, $f \in \widetilde{B}^A$ and $|||f||| \leq M$. Hence (1) holds. But then $M \leq |||f|||$ so that $M = |||f|||$.

Theorem 6. *If B is an A-Segal algebra, then $\widetilde{B}^A$ is an A-Segal algebra. Furthermore, B is a closed ideal of $\widetilde{B}^A$ with $\|f\|_B = |||f|||$ for all $f \in B$.*

Proof. We have already noted that $\widetilde{B}^A$ is an A-Segal algebra. As Theorem 5 permits a simplified proof we include the details.

Suppose $f, g \in \widetilde{B}^A$, α, β are complex numbers. Then

$$\|(\alpha f + \beta g)e\|_B \leq |\alpha|\ \|fe\|_B + |\beta|\ \|ge\|_B \leq |\alpha|\ |||f||| + |\beta|\ |||g|||.$$

Taking Sup over $e \in E$ shows that $\widetilde{B}^A$ is a normed linear space. Suppose now that $\langle f_n \rangle$ is Cauchy in $\widetilde{B}^A$. Clearly $|||\ ||| \geq \|\ \|_A$ so that $\langle f_n \rangle$ is Cauchy in A. Whence $\exists f \in A$ so that $\|f_n - f\|_A \to 0$. Given $\epsilon > 0$ choose N such that $|||f_m - f_n||| < \epsilon$ $(m, n \geq N)$. Fix $e \in E$. Now $\|(f_m - f_n)e\|_B \leq \epsilon$ and $\|(f_n - f)e\|_B \leq \|e\|_B \|f_n - f\|_A \to 0$ so that letting $n \to \infty$ in the first inequality we obtain $\|(f_m - f)e\|_B < \epsilon$ $(m \geq N)$. Thus, by Theorem 5 we get $f_m - f \in \widetilde{B}^A$. Hence $f \in \widetilde{B}^A$ and $|||f_m - f||| < \epsilon$ so that $\widetilde{B}^A$ is complete. Suppose now that $f \in A$, $g \in \widetilde{B}^A$. Then $\|fge\|_B \leq \|f\|_A \|ge\|_B$. By Theorem 6 we get $\|fge\|_B \leq \|f\|_A |||g|||$ and again by Theorem 5, $fg \in \widetilde{B}^A$. Taking Sup over $e \in E$ gives the module inequality and we have shown that $\widetilde{B}^A$ is an A-Segal algebra.

We will now show that B is a closed ideal of $\widetilde{B}^A$ and $\|f\|_B = |||f|||$ for all $f \in B$. We first show that $|||f||| = \|f\|_B$ for all $f \in B$. If $f \in B$, then $f \in \widetilde{B}^A$ so that $\|fe\|_B \leq |||f|||$ $(e \in E)$. As E is a right approximate unit we see that $\|f\|_B \leq |||f|||$. But we already know that $|||f||| \leq \|f\|_B$ for all $f \in B$ so that $\|f\|_B = |||f|||$ for all $f \in B$. This and the completeness of B (in $\|\ \|_B$) entail that B is complete with respect to $|||\ |||$. Whence B is closed in $\widetilde{B}^A$. Clearly B is an ideal of $\widetilde{B}^A$ so the proof is complete.

Our next theorem provides information about the topology of an A-Segal algebra.

Theorem 7. *Let B be an A-Segal algebra. If $n > 0$ then* $S_B(\eta) = \overline{S_B(\eta)}^A \cap B$.

Proof. It suffices to show that $\overline{S_B(\eta)}^A \cap B \subseteq S_B(\eta)$. To this end let $\eta > 0$ and $\epsilon > 0$ be given. Choose $f \in \overline{S_B(\eta)}^A \cap B$ and a sequence $\langle e_n \rangle$ from E so that $\|fe_n - f\|_B \to 0$. Now choose an integer j so that $\|fe_j - f\|_B < \epsilon/2$. As $f \in \overline{S_B(\eta)}^A$ we may choose a sequence $\langle f_n \rangle$ from $S_B(\eta)$ with $\|f_n - f\|_A \to 0$. Now, $\|f_n e_j - fe_j\|_B \leq \|f_n - f\|_A \|e_j\|_B$. Thus, with our choice of j we may choose an integer m so that $\|f_m e_j - fe_j\|_B < \epsilon/2$. We then see that $\|f_m e_j - f\|_B < \epsilon$. But $f_m e_j \in S_B(\eta)$ since $\|f_m e_j\|_B \leq \|f_m\|_B \leq \eta$. Whence $f \in S_B(\eta)$ as $S_B(\eta)$ is closed in B.

Remark 8. We point out that Theorem 7 gives another way of showing that B is closed in $\widetilde{B}^A$.

As it turns out there are many interesting cases when $B \neq \widetilde{B}^A$. Here are three examples.

Examples 9 [5]. Take $A = L^1(\mathbf{R})$ and consider the Segal algebras: (i) $B = L_A(\mathbf{R})$, the algebra of all $f \in L^1(\mathbf{R})$ with f absolutely continuous and $Df \in L^1(\mathbf{R})$. The $L_A(\mathbf{R})$ norm is given by $\|f\|_{L_A} = \|f\|_1 + \|Df\|_1$.

(ii) $B = T(\mathbf{R})$, the algebra of all continuous f with

$$\|f\|_T \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} \max_{x\in[n,n+1]} |f(x)| < \infty.$$

The $T(\mathbf{R})$ norm is $\| \ \|_T$.

In [5] we show that (i) $\widetilde{L_A(\mathbf{R})}^{L^1}$ is the algebra of all $f \in L^1(\mathbf{R})$ such that f is of bounded total variation on $\mathbf{R}$ and $|||f||| = \|f\|_1 + V_f$ where V_f is the total variation of f; (ii) $\widetilde{T(\mathbf{R})}^{L^1}$ is the algebra of all bounded $f \in L^1(\mathbf{R})$ such that

$$|||f||| = \sum_{k=-\infty}^{\infty} \operatorname{ess\,sup}_{x\in[k,k+1]} |f(x)| < \infty.$$

In both cases we have the proper inclusion $B \subsetneq \widetilde{B}^A$.

Now take $A = L^1(T)$. Define B to be the set of all $f \in L^1(T)$ for which $\|f - D_N * f\|_1 \to 0$. The B norm is defined in this way:

$$\|f\|_B = \operatorname{Sup}_{n\geq 1} \|D_N * f\|_1.$$

Here D_N denotes the Dirichlet kernel of order N. It is not too difficult to show that $\widetilde{B}^{L^1}$ consists of all $f \in L^1(T)$ for which $|||f||| = \operatorname{Sup}_{n\geq 1} \|D_N * f\|_1 < \infty$. The function $f(t) = \sum_{n=2}^{\infty} (\cos nt)/(\ln n)$ shows that $B \subsetneq \widetilde{B}^A$.

The preceding examples give rise to this question. What is the largest possible A-Segal algebra $\subsetneq A$ in which B can be embedded as a closed ideal? It will be shown in Corollary 12 that $\widetilde{B}^A$ is the maximal such algebra.

Definition 10. We say the A-Segal algebra B is singular $\Leftrightarrow$ B can be embedded as a closed (left, right, two-sided) ideal in some A-Segal algebra B' with $B \subsetneq B' \subsetneq A$. Segal algebras that are not singular are called non-singular.

Theorem 11. *If B is an A-Segal algebra, then B is singular* $\Leftrightarrow B \subsetneq \widetilde{B}^A$.

Proof. If $B \subsetneq \widetilde{B}^A$, then B is singular by Theorem 6. Suppose, conversely, that B is singular. Let B' be as specified in Definition 10. Let $f \in B'$. There is a sequence $\langle e_n \rangle$ in B such that $\|fe_n - f\|_A \to 0$. Now $\|fe_n\|_{B'} \leq \|f\|_{B'}$. But B is closed in B' so there exists $M > 0$ with $\|h\|_B \leq M\|h\|_{B'}$ for all $h \in B$. Whence, taking $h = fe_n$, we see that $\|fe_n\|_B \leq M\|fe_n\|_{B'} \leq M\|f\|_{B'}$ for all n. Thus $f \in \widetilde{B}^A$ because $\|fe_n - f\|_A \to 0$ (Lemma 4). We have shown that $B' \subseteq \widetilde{B}^A$ and so $B \subsetneq \widetilde{B}^A$ which is what we wanted to show.

Our proof of Theorem 11 shows that $\widetilde{B}^A$ is maximal among all subalgebras of A containing B as a closed ideal. Indeed,

Corollary 12. *If B' is an A-Segal algebra which contains B as a closed ideal, then $B' \subseteq \tilde{B}^A$. In particular, if $\tilde{B}^A \subseteq B'$, then $B' = \tilde{B}^A$.*

In contrast with 12 there may be many distinct ideals between B and $\tilde{B}^A$. Indeed, trivial modifications of a result due to J. Cigler [7, p. 277] obtain the following result.

Theorem 13. *If B is singular and* dimension$(\tilde{B}^A/B) \geq 2$, *then there are c closed ideals of $\tilde{B}^A$ lying properly between B and $\tilde{B}^A$. Using the Hewitt, et al., module factorization theorem one can prove*

Theorem 14. *If $f \in \tilde{B}^A$, then $f \in B \Leftrightarrow$ given any $\epsilon > 0$ there exists $e(f, \epsilon) = e \in E$ so that $|||fe - f||| < \epsilon$. Furthermore, if B' is any A-Segal algebra for which $AB' = B$, then $B' \subseteq \tilde{B}^A$.*

The following corollary is an essential tool for proofs of our ideal theorems announced in [4]. We omit the easy proof.

Corollary 15. *If E is two sided, then $A\tilde{B}^A \subseteq B$. In particular $\tilde{B}^A \cdot \tilde{B}^A \subseteq B$ so that $\tilde{B}^A$ fails to have the factorization property whenever B is singular.*

Our final result, is used in our analysis of the multiplier theory of commutative Segal algebras [5].

Corollary 16. *Let $A = L^1(G)$ where G is a locally compact group. Let $B = S(G)$ be a symmetric Segal algebra (commutative Segal algebras are symmetric). Then an $f \in \tilde{S}^{L^1}(G)$ belongs to $S(G) \Leftrightarrow |||L_y f - f|||$, $|||R_y f - f|||$ both tend to zero as y tends to the group identity.*

Proof. Symmetry entails that L_y and R_y are continuous on $S(G)$ [15, p. 17]. The same proof which shows that $S(G)$ has two-sided approximate units (of $L^1(G)$ norm 1) [15, pp. 34–37] shows that the subalgebra of $\tilde{S}^{L^1}(G)$, call it $S'(G)$, on which L_y and R_y act continuously has two sided approximate units. Whence by Theorem 14 we have $S'(G) = S(G)$. On the other hand, if $f \in S(G)$ then $\|L_y f - f\|_S \to 0$, $\|R_y f - f\|_S \to 0$ and Theorem 6 shows that $|||L_y f - f|||$, $|||R_y f - f|||_S \to 0$. The proof is complete.

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