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ITERATES OF AN ABSOLUTELY CONTINUOUS OPERATOR

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ABSTRACT. It is proved that the powers of an absolutely continuous polynomially bounded operator, the intersection of whose spectrum with the circle has zero measure, go to zero strongly. From this we deduce a simple treatment of the Nagy-Foiaş result on completely nonunitary contractions.

In this note we generalize a theorem of Nagy and Foiaş [2, Chapter II, Proposition 6.7] to polynomially bounded operators. Our method also gives an elementary proof of their theorem by avoiding the use of dilation theory completely.

Recall that a polynomially bounded operator T is said to be absolutely continuous if for every x, y in H there exists a representing measure μ for the functional $p \to (p(T)x,y)$ such that $\mu \ll m$ (= the Lebesque measure on the unit circle Γ).

Theorem. *If T is an absolutely continuous polynomially bounded operator for which $m(\sigma(T) \cap \Gamma) = 0$, then $T^n \to^{st} 0$ and $T^{*n} \to^{st} 0$.*

Proof. Let $M = \{h \in H: \|T^n h\| \to 0\}$, and let P denote the projection on $M^\perp$. If $S = PT|M^\perp$, then $S^n = PT^n|M^\perp$ for $n \geq 0$ and, hence, S is an absolutely continuous polynomially bounded operator. Moreover $\sigma(S) \cap \Gamma \subset \partial\sigma(S) \cap \Gamma \subset \sigma(T) \cap \Gamma$ and $m(\sigma(S) \cap \Gamma) = 0$. It is easy to see that for h in $M^\perp$, $\|S^n h\| \to 0$ only if $h = 0$, and since S is powerbounded, this implies that $\inf_{n \geq 0}\|S^n h\| = \mu(h)$ is strictly positive unless $h = 0$. Thus there exists a one-to-one bounded positive operator A on $M^\perp$ such that $(Ah, k) = L\{(S^n h, S^n k)\}$, where L denotes a generalized Banach limit on the set of bounded sequences. If X denotes $A^{1/2}$, then XSX^{-1} is an

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isometry on $R(X)$ and can be extended to an isometry V on $M^{\perp}$. We claim that $\sigma(V) \cap \Gamma \subset \sigma(S) \cap \Gamma$. In fact, if $\|(S-\lambda)g\| \geq C\|g\|$ for all g in $M^{\perp}$, then

$$\begin{aligned}\|(V-\lambda)Xh\|^2 &= \|(XSX^{-1}-\lambda)Xh\|^2 = \|X(S-\lambda)h\|^2 \\ &= L\{\|S^n(S-\lambda)h\|^2\} = L\{\|(S-\lambda)S^nh\|^2\} \\ &\geq L\{C^2\|S^nh\|^2\} = C^2\|Xh\|^2.\end{aligned}$$

Hence, by continuity for all g in $M^{\perp}$, we have $\|(V-\lambda)g\| \geq C\|g\|$. Thus $m(\sigma(V) \cap \Gamma) = 0$ and it follows that V is a unitary for which $m(\sigma(V)) = 0$. Since $XS^n = V^nX$ for $n \geq 0$, it follows from the F. and M. Riesz theorem [1, p.47] that the spectral measure of V is absolutely continuous. Thus we have a contradiction unless $M^{\perp} = (0)$. Since T^* satisfies the same hypothesis as T, we have $T^{*n} \xrightarrow{st} 0$.

Corollary (Nagy-Foiaş). *If T is a c. n. u. contraction satisfying $m(\sigma(T) \cap \Gamma) = 0$, then T is in C_{00}.*

Proof. We need only note that the singular part T_0 of T vanishes. But the uniqueness of the representing measures $\sigma_{x,y}$ of $P \to (P(T_0)x, y)$ allows us to extend $P \to P(T_0)$ to a representation Φ of $C(\Gamma)$, and this representation is still of norm one. Hence, $\|T_0^{-1}\| = \|\Phi(e^{-i\theta})\| \leq 1$, so that T_0 is unitary.

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