WEAKLY COMPACT GROUPS OF OPERATORS

T. A. GILLESPIE AND T. T. WEST

ABSTRACT. It is shown that the weakly closed algebra generated by a weakly compact group of operators on a Banach space is reflexive and equals its second commutant. Also, an example is given to show that the generator of a monothetic weakly compact group of operators need not have a logarithm in the algebra of all bounded linear operators on the underlying space.

Let $X$ be a complex Banach space, $B(X)$ the algebra of all bounded linear operators on $X$, and $I$ the identity operator on $X$. By a group in $B(X)$ we shall mean a multiplicative group with unit $I$. The weak operator topology on $B(X)$ is denoted by the letter $w$. Given a nonempty subset $\mathcal{E}$ of $B(X)$, $\mathcal{E}'$ and $\mathcal{E}''$ denote the first and second commutants of $\mathcal{E}$, and $A(\mathcal{E})$ is the $w$-closed subalgebra of $B(X)$ generated by $\mathcal{E}$ and $I$. The lattice of all $\mathcal{E}$-invariant closed subspaces of $X$ is denoted by $\text{Lat} \mathcal{E}$, and

$$\text{Alg Lat} \mathcal{E} = \{ T \in B(X): T(L) \subseteq L \ (L \in \text{Lat} \mathcal{E}) \}.$$ 

A subalgebra $A$ of $B(X)$ is reflexive if $\text{Alg Lat} A = A$. It is clear that reflexive algebras are $w$-closed and contain $I$. Finally, $C$, $R$, $Z$ and $T$ are the complex numbers, the reals, the integers and the unit circle.

We present several results concerning $w$-compact groups in $B(X)$. Such groups come within the general framework discussed by de Leeuw in [1], where the underlying space is called a $G$-space. The monothetic (singly generated) case has been considered in [4], [5], where an operator in $B(X)$ generating a $w$-compact group (with unit $I$) is called a $G$-operator. It was shown in [4] that, if $\mathcal{G}$ is a monothetic $w$-compact group, then $A(\mathcal{G})$ is reflexive and $\mathcal{G}'' = A(\mathcal{G})$. In fact the methods developed there and in [5] can be extended to prove

Theorem 1. Let $\mathcal{G}$ be an abelian $w$-compact group in $B(X)$ (with unit $I$). Then $A(\mathcal{G})$ is reflexive and $\mathcal{G}'' = A(\mathcal{G})$. 

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Problem A. Does Theorem 1 remain valid if the hypothesis that $\mathcal{G}$ be abelian is omitted?

$G$-operators have occurred in the work of Ljubič [8], where a study is made of the spectral properties of an operator $S \in B(X)$ satisfying $\|\exp (i r S)\| \leq M$ ($r \in \mathbb{R}$) and a certain almost-periodic condition. (The space $X$ is taken to be weakly sequentially complete.) It is shown that each such $S$ has a total set of eigenvectors corresponding to real eigenvalues, from which it is easily seen (via [4, Theorem 1.2]) that $\exp (iS)$ is a $G$-operator.

Problem B. Can every $G$-operator be written as $\exp (iS)$ for some bounded $S$?

Solution A. Theorem 1 does indeed extend to the nonabelian case and we sketch the main ideas of the proof.

Let $\mathcal{G}$ be a $\omega$-compact group in $B(X)$.

Lemma 2 [2, Theorem 8.1]. $X$ is the closed linear span of finite dimensional $\mathcal{G}$-invariant subspaces.

An easy consequence of this is the following description of $\text{Lat } \mathcal{G}$ (cf. Corollary 1.4 of [4]).

Lemma 3. Each subspace in $\text{Lat } \mathcal{G}$ is spanned by finite dimensional $\mathcal{G}$-irreducible subspaces.

Write $X^{(n)}$ for the direct sum of $n$ copies of $X$ and $T^{(n)} \in B(X^{(n)})$ for the $n$th direct sum of $T$. Putting $\mathcal{G}^{(n)} = \{T^{(n)}: T \in \mathcal{G}\}$, it is easy to see that $\mathcal{G}^{(n)}$ is a $\omega$-compact group in $B(X^{(n)})$ with unit $I^{(n)}$.

Lemma 4. Let $S \in \text{Alg Lat } \mathcal{G}$. Then $S^{(n)} \in \text{Alg Lat } \mathcal{G}^{(n)}$ for $n = 1, 2, \ldots$.

This is the key result and we sketch its proof. A straightforward argument reduces the proof to the case $n = 2$. It is then sufficient, by Lemma 3, to show that each finite dimensional $\mathcal{G}^{(2)}$-irreducible subspace $M$ of $X^{(2)}$ is $S^{(2)}$-invariant. Using irreducibility, this is easily done in the case when $M$ contains $(0, x)$ for some $x \neq 0$. Suppose therefore that $M$ contains no elements of this form. Then there is a finite dimensional subspace $N$ of $X$ and a linear operator $U: N \rightarrow X$ such that $M = \{(x, Ux): x \in N\}$. The hypotheses on $M$ imply that $N$ and $UN$ belong to $\text{Lat } \mathcal{G}$ and that $(UT - TU)(N) = \{0\}$ ($T \in \mathcal{G}$). Also, $N$ is $\mathcal{G}$-irreducible and thus either

(i) $U(N) = N$, or

(ii) $U(N) \cap N = \{0\}$. 

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(i) $U(N) = N$, or

(ii) $U(N) \cap N = \{0\}$. 

In case (i), $U$ commutes with the irreducible set of operators $\mathcal{G}|N$ on $N$. Hence $U$ is a scalar, from which it follows that $M \in \text{Lat } S^{(2)}$. In case (ii), the subspace $L = (I + U)N$ is $\mathcal{G}$- and hence $S$-invariant. Therefore, given $x \in N$, there exists $y \in N$ such that $Sx - y = -SUx + Uy$. The left-hand side of this equation is in $N$ and the right in $U(N)$, since $N$ and $U(N)$ are $S$-invariant. Therefore both sides are zero, giving $SUx = USx \ (x \in N)$, and hence $M \in \text{Lat } S^{(2)}$.

A standard argument (cf. [9, Lemma 1]) now gives

**Theorem 5.** $A(\mathcal{G})$ is reflexive.

Given $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$, the $n \times n$ complex matrices, and $x = (x_1, \ldots, x_n) \in X^{(n)}$, let $Ax$ denote the element $y = (y_1, \ldots, y_n)$ in $X^{(n)}$ defined by $y_i = \sum_{j=1}^{n} a_{ij} x_j$. Let $M$ be a finite dimensional $\mathcal{G}$-invariant subspace of $X$ with basis $\{u_1, \ldots, u_n\}$. Given $T \in \mathcal{G}$, let $Tu_i = \sum_{j=1}^{n} a_{ij}(T)u_j$.

**Lemma 6.** The map $\alpha: T \to \alpha(T) = [\alpha_{ij}(T)]$ is an anti-representation of $\mathcal{G} \to \mathbb{M}_n(\mathbb{C})$.

Define the operator $P$ in $B(X^{(n)})$ by

$$P_x = \int_{\mathcal{G}} \alpha(T^{-1})T^{(n)}x \ dT,$$

where $dT$ denotes Haar measure on $\mathcal{G}$. $P$ is a projection, but this fact is not needed here. What is needed is the following result, which is easily verified using Lemma 6.

**Lemma 7.** $PT^{(n)}x = P\alpha(T)x$ for $T \in \mathcal{G}$ and $x \in X^{(n)}$.

Defining $u$ in $X^{(n)}$ by $u = (u_1, \ldots, u_n)$, we have $T^{(n)}u = \alpha(T)u$ for each $T \in \mathcal{G}$. Therefore, from the definition of $P$, $Pu = u$. Since $u \neq 0$, it follows that $\ker P$ is strictly smaller than $X^{(n)}$. Thus, if $X^*$ is the dual space of $X$, there exists $f = (f_1, \ldots, f_n) \in X^*^{(n)}$ with $f \neq 0$ such that $f$ annihilates $\ker P$ (making the obvious identification of the dual space of $X^{(n)}$ with $X^*^{(n)}$). Put

$$F = \sum_{i=1}^{n} f_i \otimes u_i.$$

Then $F \neq 0$. Using the fact that $T^{(n)}x - \alpha(T)x$ belongs to $\ker P$ for every $T \in \mathcal{G}$ and $x \in X^{(n)}$, a routine calculation gives

**Lemma 8.** $F \in \mathcal{G}'$.

**Lemma 9.** Let $M$ be $\mathcal{G}$-irreducible and let $S \in \mathcal{G}$. Then $M \in \text{Lat } S$. 

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To see this observe that $F$ and $S$ commute. Further $\{0\} \neq F(X) \subset M$, and since $F \in \mathcal{O}_1, F(X)$ is $\mathcal{O}$-invariant. By irreducibility $F(X) = M$; but then $S(M) = SF(X) = FS(X) \subset M$.

Using reflexivity, Lemmas 9 and 3 give $\mathcal{O}'' \subset A(\mathcal{O})$. Since the reverse inclusion always holds, we have thus proved

**Theorem 10.** $\mathcal{O}'' = A(\mathcal{O})$.

**Solution B.** We give an example of a $G$-operator on a weakly sequentially complete space which is not of the form $\exp(iS)$ with $S$ bounded. This example depends on some general facts about logarithms of point measures.

Let $G$ be a LCAG and let $M(G)$ be the commutative Banach algebra of bounded regular complex measures on $G$ under convolution. Given $x \in G$, we show that the point mass $\delta_x$ has a logarithm in $M(G)$ if, and only if, $x$ is of finite order in $G$. The "if" proof follows from elementary spectral theory. For the converse, let $M_d(G)$ be the Banach algebra of discrete measures on $G$.

**Lemma 11.** If a discrete measure $\mu$ on $G$ has a logarithm in $M(G)$, then $\mu$ has a logarithm in $M_d(G)$.

This follows from the fact that if $\mu = \exp \nu$ for some $\nu \in M(G)$, then $\mu = \exp \nu_d$ where $\nu_d$ is the discrete part of $\nu$.

**Lemma 12.** Let $\delta_x = \exp \nu$ in $M(G)$. Then $x$ is of finite order in $G$.

By Lemma 11 we may (and do) assume that $G$ is discrete. Then the maximal ideal space of $M(G)$ is the compact group $\hat{G}$ dual to $G$. The proof can be completed by the following simple argument due to Gavin Brown.

Taking Gelfand transforms in the equation $\delta_x = \exp \nu$, we obtain

$$x(\chi) = \delta_x(\chi) = \exp \hat{\nu}(\chi) \quad (\chi \in \hat{G}),$$

where, without loss of generality, $\hat{\nu}(\chi_1) = 0$ for $\chi_1$ the unit of $\hat{G}$. Since $x$ is a character on $\hat{G}$, it follows that

$$\hat{\nu}(\chi \psi) = \hat{\nu}(\chi) + \hat{\nu}(\psi) + 2\pi i N(\chi, \psi) \quad (\chi, \psi \in \hat{G})$$

where $N: \hat{G} \times \hat{G} \rightarrow \mathbb{Z}$ is continuous. Let $H$ be the connected component containing $\chi_1$ in $\hat{G}$. Then $2\pi i N(\chi, \psi) = -\hat{\nu}(\chi_1) = 0$ on $H$. Hence

$$\hat{\nu}(\chi^n) = n\hat{\nu}(\chi) \quad (\chi \in H, n \in \mathbb{Z}).$$

The boundedness of the continuous function $\hat{\nu}$ on the compact group $H$
gives $\hat{\nu}(\chi) = 0 \ (\chi \in H)$. Hence $x(\chi) = 1 \ (\chi \in H)$ and so $x$ is of finite order in $G$ [6, 24.20].

We can now give the counterexample for Problem B. Let $R_\omega$ be the translation operator on $L^1(T)$ defined by

$$(R_\omega f)(t) = f(t\omega^{-1}) \quad (f \in L^1(T), \ t \text{ a.e.}),$$

where $\omega \in T$ and $\arg \omega$ is an irrational multiple of $2\pi$. $R_\omega$ is a $G$-operator on $L^1(T)$ [4, Example 5.4] and $L^1(T)$ is weakly sequentially complete [3, IV. 8.6].

**Theorem 13.** $R_\omega$ does not have a logarithm in $B(L^1(T))$.

For suppose $R_\omega = \exp S$ in $B(L^1(T))$. Since the powers of $\omega$ are dense in $T$, it follows that $S$ commutes with every translation $R_t \ (t \in T)$. Hence $S$ is a multiplier on $L^1(T)$ and there exists $\mu \in M(T)$ such that $Sf = \mu * f \ (f \in L^1(T))$ [7, Theorem 0.1.1]. Therefore

$$\delta_\omega * f = R_\omega f = (\exp \mu) * f \quad (f \in L^1(T))$$

and so $\delta_\omega = \exp \mu$. Lemma 12 gives the required contradiction.

**REFERENCES**