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WEAKLY COMPACT GROUPS OF OPERATORS

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ABSTRACT. It is shown that the weakly closed algebra generated by a weakly compact group of operators on a Banach space is reflexive and equals its second commutant. Also, an example is given to show that the generator of a monothetic weakly compact group of operators need not have a logarithm in the algebra of all bounded linear operators on the underlying space.

Let X be a complex Banach space, $B(X)$ the algebra of all bounded linear operators on X, and I the identity operator on X. By a group in $B(X)$ we shall mean a multiplicative group with unit I. The weak operator topology on $B(X)$ is denoted by the letter w. Given a nonempty subset $\mathfrak{S}$ of $B(X)$, $\mathfrak{S}'$ and $\mathfrak{S}''$ denote the first and second commutants of $\mathfrak{S}$, and $A(\mathfrak{S})$ is the w-closed subalgebra of $B(X)$ generated by $\mathfrak{S}$ and I. The lattice of all $\mathfrak{S}$-invariant closed subspaces of X is denoted by Lat $\mathfrak{S}$, and

$$\text{Alg Lat } \mathfrak{S} = \{T \in B(X)\colon T(L) \subset L \ \ (L \in \text{Lat } \mathfrak{S})\}.$$

A subalgebra A of $B(X)$ is *reflexive* if Alg Lat $A = A$. It is clear that reflexive algebras are w-closed and contain I. Finally, $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{T}$ are the complex numbers, the reals, the integers and the unit circle.

We present several results concerning w-compact groups in $B(X)$. Such groups come within the general framework discussed by de Leeuw in [1], where the underlying space is called a *G-space*. The monothetic (singly generated) case has been considered in [4], [5], where an operator in $B(X)$ generating a w-compact group (with unit I) is called a *G-operator*. It was shown in [4] that, if $\mathfrak{G}$ is a monothetic w-compact group, then $A(\mathfrak{G})$ is reflexive and $\mathfrak{G}'' = A(\mathfrak{G})$. In fact the methods developed there and in [5] can be extended to prove

Theorem 1. *Let $\mathfrak{G}$ be an abelian w-compact group in $B(X)$ (with unit I). Then $A(\mathfrak{G})$ is reflexive and $\mathfrak{G}'' = A(\mathfrak{G})$.*

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Problem A. Does Theorem 1 remain valid if the hypothesis that $\mathcal{G}$ be abelian is omitted?

G-operators have occurred in the work of Ljubič [8], where a study is made of the spectral properties of an operator $S \in B(X)$ satisfying $\|\exp(i\,r\,S)\| \leq M$ $(r \in \mathbf{R})$ and a certain almost-periodic condition. (The space X is taken to be weakly sequentially complete.) It is shown that each such S has a total set of eigenvectors corresponding to real eigenvalues, from which it is easily seen (via [4, Theorem 1.2]) that $\exp(iS)$ is a G-operator.

Problem B. Can every G-operator be written as $\exp(iS)$ for some bounded S?

Solution A. Theorem 1 does indeed extend to the nonabelian case and we sketch the main ideas of the proof.

Let $\mathcal{G}$ be a w-compact group in $B(X)$.

Lemma 2 [2, Theorem 8.1]. *X is the closed linear span of finite dimensional $\mathcal{G}$-invariant subspaces.*

An easy consequence of this is the following description of Lat $\mathcal{G}$ (cf. Corollary 1.4 of [4]).

Lemma 3. *Each subspace in* Lat $\mathcal{G}$ *is spanned by finite dimensional $\mathcal{G}$-irreducible subspaces.*

Write $X^{(n)}$ for the direct sum of n copies of X and $T^{(n)} \in B(X^{(n)})$ for the nth direct sum of T. Putting $\mathcal{G}^{(n)} = \{T^{(n)}: T \in \mathcal{G}\}$, it is easy to see that $\mathcal{G}^{(n)}$ is a w-compact group in $B(X^{(n)})$ with unit $I^{(n)}$.

Lemma 4. *Let $S \in$ Alg Lat $\mathcal{G}$. Then $S^{(n)} \in$ Alg Lat $\mathcal{G}^{(n)}$ for $n = 1, 2, \cdots$.*

This is the key result and we sketch its proof. A straightforward argument reduces the proof to the case $n = 2$. It is then sufficient, by Lemma 3, to show that each finite dimensional $\mathcal{G}^{(2)}$-irreducible subspace M of $X^{(2)}$ is $S^{(2)}$-invariant. Using irreducibility, this is easily done in the case when M contains $(0, x)$ for some $x \neq 0$. Suppose therefore that M contains no elements of this form. Then there is a finite dimensional subspace N of X and a linear operator $U: N \to X$ such that $M = \{(x, Ux): x \in N\}$. The hypotheses on M imply that N and UN belong to Lat $\mathcal{G}$ and that $(UT - TU)(N) = \{0\}$ $(T \in \mathcal{G})$. Also, N is $\mathcal{G}$-irreducible and thus either

(i) $U(N) = N$, or

(ii) $U(N) \cap N = \{0\}$.

In case (i), U commutes with the irreducible set of operators $\mathcal{G}|N$ on N. Hence U is a scalar, from which it follows that $M \in \operatorname{Lat} S^{(2)}$. In case (ii), the subspace $L = (I + U)N$ is $\mathcal{G}$- and hence S-invariant. Therefore, given $x \in N$, there exists $y \in N$ such that $Sx - y = -SUx + Uy$. The left-hand side of this equation is in N and the right in $U(N)$, since N and $U(N)$ are S-invariant. Therefore both sides are zero, giving $SUx = USx$ $(x \in N)$, and hence $M \in \operatorname{Lat} S^{(2)}$.

A standard argument (cf. [9, Lemma 1]) now gives

Theorem 5. *$A(\mathcal{G})$ is reflexive.*

Given $A = [a_{ij}] \in \mathfrak{M}_n(\mathbf{C})$, the $n \times n$ complex matrices, and $\mathbf{x} = (x_1, \cdots, x_n) \in X^{(n)}$, let $A\mathbf{x}$ denote the element $\mathbf{y} = (y_1, \cdots, y_n)$ in $X^{(n)}$ defined by $y_i = \sum_{j=1}^n a_{ij} x_j$. Let M be a finite dimensional $\mathcal{G}$-invariant subspace of X with basis $\{u_1, \cdots, u_n\}$. Given $T \in \mathcal{G}$, let $Tu_i = \sum_{j=1}^n \alpha_{ij}(T) u_j$.

Lemma 6. *The map $\alpha: T \to \alpha(T) = [\alpha_{ij}(T)]$ is an anti-representation of $\mathcal{G} \to \mathfrak{M}_n(\mathbf{C})$.*

Define the operator $\mathbf{P}$ in $B(X^{(n)})$ by

$$\mathbf{P}\mathbf{x} = \int_{\mathcal{G}} \alpha(T^{-1}) T^{(n)} \mathbf{x} \, dT,$$

where dT denotes Haar measure on $\mathcal{G}$. $\mathbf{P}$ is a projection, but this fact is not needed here. What is needed is the following result, which is easily verified using Lemma 6.

Lemma 7. $\mathbf{P}T^{(n)}\mathbf{x} = \mathbf{P}\alpha(T)\mathbf{x}$ *for* $T \in \mathcal{G}$ *and* $\mathbf{x} \in X^{(n)}$.

Defining $\mathbf{u}$ in $X^{(n)}$ by $\mathbf{u} = (u_1, \cdots, u_n)$, we have $T^{(n)}\mathbf{u} = \alpha(T)\mathbf{u}$ for each $T \in \mathcal{G}$. Therefore, from the definition of $\mathbf{P}$, $\mathbf{P}\mathbf{u} = \mathbf{u}$. Since $\mathbf{u} \neq \mathbf{0}$, it follows that $\ker \mathbf{P}$ is strictly smaller than $X^{(n)}$. Thus, if X^* is the dual space of X, there exists $\mathbf{f} = (f_1, \cdots, f_n) \in X^{*(n)}$ with $\mathbf{f} \neq \mathbf{0}$ such that $\mathbf{f}$ annihilates $\ker \mathbf{P}$ (making the obvious identification of the dual space of $X^{(n)}$ with $X^{*(n)}$). Put

$$F = \sum_{i=1}^{n} f_i \otimes u_i.$$

Then $F \neq 0$. Using the fact that $T^{(n)}\mathbf{x} - \alpha(T)\mathbf{x}$ belongs to $\ker \mathbf{P}$ for every $T \in \mathcal{G}$ and $\mathbf{x} \in X^{(n)}$, a routine calculation gives

Lemma 8. $F \in \mathcal{G}'$.

Lemma 9. *Let M be $\mathcal{G}$-irreducible and let $S \in \mathcal{G}''$. Then* $M \in \operatorname{Lat} S$.

To see this observe that F and S commute. Further $\{0\} \neq F(X) \subset M$, and since $F \in \mathcal{G}'$, $F(X)$ is $\mathcal{G}$-invariant. By irreducibility $F(X) = M$; but then $S(M) = SF(X) = FS(X) \subset M$.

Using reflexivity, Lemmas 9 and 3 give $\mathcal{G}'' \subset A(\mathcal{G})$. Since the reverse inclusion always holds, we have thus proved

Theorem 10. $\mathcal{G}'' = A(\mathcal{G})$.

Solution B. We give an example of a G-operator on a weakly sequentially complete space which is not of the form $\exp(iS)$ with S bounded. This example depends on some general facts about logarithms of point measures.

Let G be a LCAG and let $M(G)$ be the commutative Banach algebra of bounded regular complex measures on G under convolution. Given $x \in G$, we show that the point mass δ_x has a logarithm in $M(G)$ if, and only if, x is of finite order in G. The "if" proof follows from elementary spectral theory. For the converse, let $M_d(G)$ be the Banach algebra of discrete measures on G.

Lemma 11. *If a discrete measure μ on G has a logarithm in $M(G)$, then μ has a logarithm in $M_d(G)$.*

This follows from the fact that if $\mu = \exp \nu$ for some $\nu \in M(G)$, then $\mu = \exp \nu_d$ where ν_d is the discrete part of ν.

Lemma 12. *Let $\delta_x = \exp \nu$ in $M(G)$. Then x is of finite order in G.*

By Lemma 11 we may (and do) assume that G is discrete. Then the maximal ideal space of $M(G)$ is the compact group $\hat{G}$ dual to G. The proof can be completed by the following simple argument due to Gavin Brown.

Taking Gelfand transforms in the equation $\delta_x = \exp \nu$, we obtain

$$x(\chi) = \hat{\delta}_x(\chi) = \exp \hat{\nu}(\chi) \qquad (\chi \in \hat{G}),$$

where, without loss of generality, $\hat{\nu}(\chi_1) = 0$ for χ_1 the unit of $\hat{G}$. Since x is a character on $\hat{G}$, it follows that

$$\hat{\nu}(\chi\psi) = \hat{\nu}(\chi) + \hat{\nu}(\psi) + 2\pi i N(\chi, \psi) \qquad (\chi, \psi \in \hat{G})$$

where $N: \hat{G} \times \hat{G} \to \mathbf{Z}$ is continuous. Let H be the connected component containing χ_1 in $\hat{G}$. Then $2\pi i N(\chi, \psi) = -\hat{\nu}(\chi_1) = 0$ on H. Hence

$$\hat{\nu}(\chi^n) = n\hat{\nu}(\chi) \qquad (\chi \in H, n \in \mathbf{Z}).$$

The boundedness of the continuous function $\hat{\nu}$ on the compact group H

gives $\hat{\nu}(\chi) = 0$ $(\chi \in H)$. Hence $x(\chi) = 1$ $(\chi \in H)$ and so x is of finite order in G [6, 24.20].

We can now give the counterexample for Problem B. Let R_ω be the translation operator on $L^1(\mathbf{T})$ defined by

$$(R_\omega f)(t) = f(t\omega^{-1}) \qquad (f \in L^1(\mathbf{T}),\ t \text{ a.e.}),$$

where $\omega \in \mathbf{T}$ and $\arg \omega$ is an irrational multiple of 2π. R_ω is a G-operator on $L^1(\mathbf{T})$ [4, Example 5.4] and $L^1(\mathbf{T})$ is weakly sequentially complete [3, IV. 8.6].

Theorem 13. *R_ω does not have a logarithm in $B(L^1(\mathbf{T}))$.*

For suppose $R_\omega = \exp S$ in $B(L^1(\mathbf{T}))$. Since the powers of ω are dense in $\mathbf{T}$, it follows that S commutes with every translation R_t $(t \in \mathbf{T})$. Hence S is a multiplier on $L^1(\mathbf{T})$ and there exists $\mu \in M(\mathbf{T})$ such that $Sf = \mu * f$ $(f \in L^1(\mathbf{T}))$ [7, Theorem 0.1.1]. Therefore

$$\delta_\omega * f = R_\omega f = (\exp \mu) * f \qquad (f \in L^1(\mathbf{T}))$$

and so $\delta_\omega = \exp \mu$. Lemma 12 gives the required contradiction.

REFERENCES

1. K. de Leeuw, *Linear spaces with a compact group of operators*, Illinois J. Math. 2 (1958), 367–377. MR 21 #819.

2. K. de Leeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. 105 (1961), 63–97. MR 24 #A1632.

3. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.

4. T. A. Gillespie and T. T. West, *Operators generating weakly compact groups*, Indiana Univ. Math. J. 21 (1972), 671–688.

5. ———, *Operators generating weakly compact groups*. II, Proc. Royal Irish Acad. 73A (1973), 309–326.

6. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. 1: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR 28 #158.

7. R. Larsen, *An introduction to the theory of multipliers*, Die Grundlehren der math. Wissenschaften, Band 175, Springer-Verlag, Berlin, 1971.

8. Ju. I. Ljubič, *Almost periodic functions in the spectral analysis of operators*, Dokl. Akad. Nauk SSSR 132 (1960), 518–520 = Soviet Math. Dokl. 1 (1960), 593–595. MR 22 #9863.

9. H. Radjavi and P. Rosenthal, *On invariant subspaces and reflexive algebras*, Amer. J. Math. 91 (1969), 683–692. MR 40 #4796.

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