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A SUFFICIENT CONDITION FOR THE INSERTION OF A CONTINUOUS FUNCTION

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ABSTRACT. A sufficient condition for the insertion of a continuous function between two comparable real-valued functions is given. Four insertion theorems are obtained as corollaries.

1. **Introduction.** Results of Katětov concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks, are used in order to give a sufficient condition for the insertion of a continuous function between two comparable real-valued functions. If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X. The abbreviations lsc and usc are used throughout the paper for lower semicontinuous and upper semicontinuous, respectively. The closure of a set S is denoted by $\mathrm{cl}(S)$, and $\mathrm{int}(S)$ denotes the interior of S.

2. **The main result.** Katětov [3] and Tong [6] independently extended Hahn's insertion property to normal spaces. They proved that if the space X is normal and if g and f are real-valued functions with g usc, f lsc, and $g \leq f$, then there exists a continuous function h on X such that $g \leq h \leq f$. The following theorem gives sufficient conditions, which are expressible simply in terms of indefinite cut sets and strong binary relations, for the insertability of a continuous function. The result of Katětov and Tong is a corollary of this theorem.

Theorem 1. *Let g and f be real-valued functions on X such that $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain*

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of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$, then there exists a continuous function h defined on X such that $g \leq h \leq f$.

Before giving the proof of this theorem, the necessary definitions and terminology are stated. The first two definitions are due to, or are modifications of, conditions considered in [3], [4].

Definition. If ρ is a binary relation in a set S then $\overline{\rho}$ is defined as follows: $x\ \overline{\rho}\ y$ if and only if $y\ \rho\ v$ implies $x\ \rho\ v$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.

Definition. A binary relation ρ in the power set $P(X)$ of a space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

(1) If $A_i\ \rho\ B_j$ for any i in $\{1, \cdots, m\}$ and for any j in $\{1, \cdots n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any i in $\{1, \cdots, m\}$ and any j in $\{1, \cdots, n\}$.

(2) If $A \subset B$, then $A\ \overline{\rho}\ B$.

(3) If $A\ \rho\ B$, then $\mathrm{cl}(A) \subset B$ and $A \subset \mathrm{int}(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [1] as follows:

Definition. If f is a real-valued function defined on a space X and if $\{x | f(x) < l\} \subset A(f, l) \subset \{x | f(x) \leq l\}$ for a real number l, then $A(f, l)$ is a *lower indefinite cut set* in the domain of f at the level l.

We now give the proof of Theorem 1.

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$. Define functions F and G mapping the rational numbers Q into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then $F(t_1)\ \overline{\rho}\ F(t_2)$, $G(t_1)\ \overline{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [4] it follows that there exists a function H mapping Q into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2)$, $H(t_1)\ \rho\ H(t_2)$, and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \text{glb}\{t \in Q | x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

The proof is concluded by showing that h is continuous at an arbitrary point x in X: Choose t_1 and t_2 such that $t_1 < h(x) < t_2$. If $t_1 < t < h(x)$, then $x \notin H(t)$; since $H(t_1)\ \rho\ H(t)$ implies that $\text{cl}(H(t_1)) \subset H(t)$, it follows that $x \notin \text{cl}(H(t_1))$. If $h(x) < t < t_2$, then $x \in H(t)$; since $H(t)\ \rho\ H(t_2)$ implies that $H(t) \subset \text{int}(H(t_2))$, it follows that $x \in \text{int}(H(t_2))$. Since x is in the open set $\text{int}(H(t_2)) - \text{cl}(H(t_1))$, and since any point y in this open set must satisfy $t_1 \leq h(y) \leq t_2$, it follows that h is continuous.

The above proof uses the technique of Theorem 1 of [3].

3. Applications.

The first application of Theorem 1 is the insertion theorem of Katětov and Tong which was mentioned above.

Corollary 1 [3], [6]. *If f and are real-valued functions defined on a normal space X such that f is lsc, g is usc, and $g \leq f$, then there exists a continuous function h defined on X such that $g \leq h \leq f$.*

Proof. Let g be usc, let f be lsc, and suppose that $g \leq f$. If a binary relation ρ is defined by $A\ \rho\ B$ in case $\text{cl}(A) \subset \text{int}(B)$, and if X is normal, then ρ is a strong binary relation in the power set of X. For each t in Q let $A(f, t)$ and $A(g, t)$ be any lower indefinite cut sets for f and g respectively. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

$$A(f, t_1) \subset \{x | f(x) \leq t_1\} \subset \{x | g(x) < t_2\} \subset A(g, t_2);$$

since $\{x | f(x) \leq t_1\}$ is closed and since $\{x | g(x) < t_2\}$ is open, it follows that $\text{cl}(A(f, t_1)) \subset \text{int}(A(g, t_2))$. Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho$ $A(g, t_2)$. The proof follows from Theorem 1.

The second corollary of Theorem 1 concerns normal semicontinuous functions. If f_* denotes the lower limit function of f and if f^* denotes the upper limit function of f, then f is normal lsc in case $f = (f^*)_*$ and f is normal usc in case $f = (f_*)^*$ [2]. See [2] for amplification and for alternate characterizations of normal semicontinuous functions.

Corollary 2 [5]. *Let g and f be real-valued functions defined on a space X such that g is normal usc, f is normal lsc, and $g \leq f$. If disjoint regular closed subsets of X are separated by disjoint open sets, then there exists a continuous function h on X such that $g \leq h \leq f$.*

Proof. Let the binary relation ρ be defined by $A \rho B$ in case $\mathrm{cl}(A) \subset F \subset G \subset \mathrm{int}(B)$ for some regular closed set F and for some regular open set G. Since disjoint regular closed subsets of X are separated by disjoint open sets, it can be verified that ρ is a strong binary relation in the power set of X. For each rational number t, let $A(f, t) = \{x \in X \mid f(x) < t\}$ and $A(g, t) = \{x \in X \mid g(x) \leq t\}$. Since f and g are normal semicontinuous functions, $\mathrm{cl}(A(f, t))$ is regular closed and $\mathrm{int}(A(g, t))$ is regular open. Thus if t_1 and t_2 are any elements of Q with $t_1 < t_2$, it follows that $A(f, t_1) \rho A(g, t_2)$. By Theorem 1, there exists a continuous function h defined on X such that $g \leq h \leq f$.

The proofs of the next two corollaries are omitted. In either case, the binary relation $A \rho B$ if and only if $\mathrm{cl}(A) \subset G \subset \mathrm{cl}(G) \subset \mathrm{int}(B)$ for some open set G may be used.

Corollary 3 [5]. *Let g and f be real-valued functions on a space X such that g is usc, f is lsc and at least one of f and g is normal. If disjoint closed subsets of X, at least one of which is regular closed, are separated by open sets, then there exists a continuous function h on X such that $g \leq h \leq f$.*

Corollary 4. *Let g and f be real-valued functions on a space X such that g is lsc, f is usc, and $g \leq f$. If X is extremally disconnected, then there exists a continuous function h on X such that $g \leq h \leq f$.*

We conclude with the observation that in each of the preceding corollaries, the separation property for X, which is used in order to verify that ρ is a strong binary relation, is also a necessary condition for the stated insertion property. For example, if for every lsc function g and every usc function f such that $g \leq f$ there exists a continuous function h on X with $g \leq h \leq f$, then the space X is extremally disconnected.

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