LOCAL SECTIONS OF FLOWS ON MANIFOLDS

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ABSTRACT. We show the existence of local sections for dynamical systems on manifolds which are themselves generalized manifolds of co-dimension one. One implication of this is that the level surfaces of Lyapunov functions on three-manifolds are two-manifolds. This fact is used to study the geometry of stable attractors in three-manifolds.

A dynamical system is a triple (X, R, π); X = any topological space, R = reals, and π is a continuous map: $X \times R \longrightarrow X$, for every $x \in X$, $\pi(x, 0) = x$ and $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2)$. Let $S \subset X$ and $T \subset R$; we denote $\pi(S \times T)$ as $S\pi T$ whether S, T are sets or points. A local section $S \subset X$ of extent ϵ is any set such that $S \times (-\epsilon, \epsilon)$ is topologically imbedded in X by π. For definitions of other concepts such as rest point, attractor, stable sets, etc. the reader is referred to [1].

Lemma 1. *Let X be a Tychonov space. If $p \in X$ is not a rest point for a dynamical system (X, R, π), then there is a local section S of extent $\epsilon > 0$ such that $p \in S$ and $S\pi(-\epsilon, \epsilon)$ is open in X.*

Proof. By [5, pp. 144–145] there is a local section T of extent 2ϵ for (X, R, π), containing p, such that p is interior to $T\pi(-\epsilon, \epsilon)$. An open set U about p may be chosen such that $\overline{U}$ is interior to $T\pi(-\epsilon, \epsilon)$ and hence $\overline{U}\pi(-\epsilon, \epsilon)$ is interior to $T\pi(-2\epsilon, 2\epsilon)$. We set $S = T \cap U$; clearly S is a local section of extent 2ϵ.

If $S\pi(-\epsilon, \epsilon)$ is not open in X, there is a net $\{x_\alpha\}$ in $T\pi(-2\epsilon, 2\epsilon)\backslash S\pi(-\epsilon, \epsilon)$ which converges to $x \in S\pi(-\epsilon, \epsilon)$. Let $\tau: T\pi(-2\epsilon, 2\epsilon) \longrightarrow (-2\epsilon, 2\epsilon)$ be the "time of return to T" map; i.e., for $y \in T\pi(-2\epsilon, 2\epsilon)$, $y\pi\tau(y) \in T$. Because τ is π^{-1} followed by projection onto the second factor, followed by multiplication in $(-2\epsilon, 2\epsilon)$ by -1, τ is continuous.

Now $|\tau(x)| < \epsilon$, so without loss of generality, we assume also that $|\tau(x_\alpha)| < \epsilon$ for every α. Thus $x_\alpha \in (T\backslash S)\pi(-\epsilon, \epsilon)$ for every α, so $x_\alpha\pi\tau(x_\alpha) \in$

$T \backslash S$ must converge to $x\pi\tau(x)$ in S. This is not possible, as S is open relative to T.   **

We recall the concept of a generalized manifold due to Wilder [10]. In particular, classical Euclidean n-manifolds are n-dimensional generalized manifolds (n-g.m.'s). Conversely, noncompact, connected, metric k-g.m.'s are classical k-manifolds for $k = 1, 2$.

Theorem 1. *Let X be a Euclidean n-manifold, and (X, R, π) a dynamical system on X. Then the set S of Lemma 1 is an $(n - 1)$-generalized manifold.*

Proof. The set $S\pi(-\epsilon, \epsilon)$ is an open subset of an n-manifold, and therefore an n-manifold itself. The set $S \times (-\epsilon, \epsilon)$ is then an n-manifold. By Theorem 6 of [7], we may conclude that S is itself a generalized manifold. Raymond proves, roughly, that $M \approx A \times B$ is a generalized n-manifold iff A and B are k- and r-manifolds respectively, where $n = k + r$.   **

Corollary 1. *If (X, R, π) is a dynamical system on an n-manifold X, and K is any compact local section of extent $2\epsilon > 0$ in X, then K lies in an $(n - 1)$-g.m. S which is a local section of extent ϵ. Moreover, if K is contained in an open set W, S may also be assumed to lie in W.*

Proof. By results of Hájek [5, pp. 144–145], a local section T may be found, of extent 2ϵ, such that $K \subset \text{int}(T\pi(-\epsilon, \epsilon)) \subset W\pi(-\epsilon, \epsilon)$. The procedure of Lemma 1 may then be followed to produce S, which will be an $(n - 1)$-g.m.   **

As is well known, a continuous flow on a 2-manifold has local sections which are 1-manifolds [5]. Though his terminology was different, it was known to Whitney [9] that for a nonrest point of a flow in E^3, local sections which were discs existed.

Corollary 2. *If (X, R, π) is a dynamical system on a 3-manifold X, and T is a local section of extent $\epsilon > 0$ containing a compact set K, and $K \subset \text{int } T\pi(-\epsilon, \epsilon)$, then there is a neighborhood of K in T which is a 2-manifold.*

Proof. The 2-g.m. S that is obtained in this setting is a classical 2-manifold [10].   **

Corollary 3. *If (X, R, π) is a dynamical system on a 3-manifold with boundary ∂X, and p is a nonrest point of π, then there is a 2-cell C which is a local section of extent $\epsilon > 0$, $p \in C$, and p is interior to $C\pi(-\epsilon, \epsilon)$.*

Proof. If $p \notin \partial X$, then the conclusion follows from Corollary 2. We extend the dynamical system π on X to a system on $D(X)$, the double of X.

Let $h_1 : X \twoheadrightarrow X_1$ and $h_2 : X \twoheadrightarrow X_2$ be homeomorphisms, and let π_1 and π_2 be the corresponding dynamical systems induced by h_1 and h_2 from π. Let $X_1 \cup X_2$ denote the disjoint union of X_1 and X_2, and let $q : X_1 \cup X_2 \twoheadrightarrow D(X)$ be the quotient map which identifies $p \in X_1$ and $q \in X_2$ iff there is an $x \in \partial X$ with $h_1(x) = p$, $h_2(x) = q$. $D(X)$, equipped with the quotient topology, is a 3-manifold without boundary.

We define $\widetilde{\pi} : D(X) \times R \rightarrow D(X)$ by

$$\widetilde{\pi}(x, \, t) = \begin{cases} q \circ \pi_1(q^{-1}(x), \; t) & \text{if } x \in q(X_1) \\ q \circ \pi_2(q^{-1}(x), \; t) & \text{if } x \in q(X_2) \end{cases}.$$

Since these maps agree on $q(X_1) \cap q(X_2)$, $\widetilde{\pi}$ is a dynamical system. If $p \in \partial X$ is a nonrest point under π, then $p' = q(p)$ is a nonrest point of $\widetilde{\pi}$. Because ∂X_1 is an invariant 2-manifold under π_1, $q(\partial X_1)$ is an invariant 2-manifold under $\widetilde{\pi}$. Accordingly there is an arc A in $q(\partial X_1)$ which contains p', is a section of extent $2\epsilon > 0$ in $q(\partial X_1)$, and such that p' is interior, in $q(\partial X_1)$, to $A\widetilde{\pi}(-\epsilon, \epsilon)$.

Let S be a 2-manifold local section of extent ϵ for $\widetilde{\pi}$ in $D(X)$, such that $A \subset S$, and $A \subset \text{int } S\pi(-\epsilon, \epsilon)$. In S, let B be a small 2-cell about p' whose boundary intersects both components of $A\backslash\{p'\}$. Now $A \cap \partial B$ contains points r, s, such that the subarc $\widehat{rs}$ of A contains p' and the open arc $\widehat{rs}$ does not meet ∂B. Let C' denote the component of $B\backslash\widehat{rs}$ in $q(X_1)$. Then $\text{cl}(C')$ is a 2-cell in $q(X_1)$. We define $C = h_1^{-1} \circ q^{-1}(\text{cl}(C'))$. **

Lyapunov functions. Let (X, R, π) be a dynamical system. A set $K \subset X$ is asymptotically stable if and only if there is a continuous function F, defined on an open subset containing K, with range in $[0, \infty)$ such that $F(x) = 0$ exactly when $x \in K$, and for $x \in \text{dom}(F)\backslash K$, and $t > 0$, $F(x) > F(x\pi t)$. F is called a Lyapunov function for K. The level surfaces of F are local sections of extent $+\infty$ for π, if X is locally compact and K is compact.

Theorem 2. *Let (X, R, π) be a dynamical system on an n-manifold X, and suppose that the compact set $K \subset X$ is asymptotically stable with Lyapunov function F. Then there is an $\epsilon > 0$, such that for all c, $0 < c < \epsilon$, $F^{-1}(c)$ is a compact $(n - 1)$-g.m. and a local section of extent $+\infty$.*

Proof. Let us choose $\epsilon > 0$ such that $F^{-1}[0, \epsilon]$ is a compact neighborhood of K. For fixed $c \in (0, \epsilon)$, choose δ such that $0 < c - \delta < c + \delta <$

ϵ. Then $F^{-1}(c)$ is a compact local section. Define

$$\sigma = \sup\{F(x\pi(\delta/3)): x \in F^{-1}(c)\} \quad \text{and} \quad \tau = \inf\{F(x\pi(-\delta/3)): x \in F^{-1}(c)\}.$$

Then $\sigma < c < \tau$, $F^{-1}(\sigma, \tau) \subset F^{-1}(c)\pi(-\delta/2, \delta/2)$, and $F^{-1}(c)$ is interior to $F^{-1}(\sigma, \tau)$ because F is continuous. Now by Corollary 1, there is a local section T of extent $\delta/2$, with $F^{-1}(c) \subset T \subset F^{-1}(\sigma, \tau)$, and T is an $(n-1)$-g.m. We claim that $F^{-1}(c) = T$. If $p \in T\backslash F^{-1}(c)$, then

$$(p, 0) \in (T\backslash F^{-1}(c)) \times (-\delta/2, \delta/2) = T \times (-\delta/2, \delta/2)\backslash F^{-1}(c) \times (-\delta/2, \delta/2).$$

Because the restriction of π to $T \times (-\delta/2, \delta/2)$ is injective,

$$(T\backslash F^{-1}(c))\pi(-\delta/2, \delta/2) = T\pi(-\delta/2, \delta/2)\backslash F^{-1}(c)\pi(-\delta/2, \delta/2).$$

Therefore $\pi(p, 0) = p \in T \subset T\pi(-\delta/2, \delta/2)\backslash F^{-1}(c)\pi(-\delta/2, \delta/2)$. This last inclusion is not possible because $T \subset F^{-1}(\sigma, \tau) \subset F^{-1}(c)\pi(-\delta/2, \delta/2)$.

Therefore $F^{-1}(c)$ is an $(n-1)$-g.m. **

Definition. Let X be a subspace of a topological space Y. If there is a homeomorphism h carrying $X \times (-1, 1)$ onto an open neighborhood of X in Y, such that $h(x, 0) = x$ for every $x \in X$, then X is bicollared in Y.

Lemma 2. *If (X, R, π) is a dynamical system on an n-manifold X, and K is a compact local section for (X, R, π), then K is contained in an $(n-1)$ g.m. local section S which is bicollared in X.*

Proof. In proving Corollary 1 one obtains a locally compact set S such that $K \subset S$, for some $\epsilon > 0$, S is a local section of extent ϵ, and $S\pi(-\epsilon, \epsilon)$ is open in X. Therefore π maps $S \times (-\epsilon, \epsilon)$ homeomorphically onto an open subset of X, so S is bicollared. It is proven in Theorem 1 that S is an $(n-1)$-g.m. **

From Lemma 2 and Theorem 2, we can prove the following:

Corollary 4. *If X in Theorem 2 is a 3-manifold, then $F^{-1}(c)$ is a compact bicollared 2-manifold in X.* **

In what follows, we use Čech homology with integral coefficients.

Theorem 3. *Let $X = S^3$ or R^3, and let K be any compact, connected, nonseparating subset of X with $H_1(K) = 0$. If K is asymptotically stable for (X, R, π), then any Lyapunov function for K has level surfaces which are 2-spheres, bicollared in X.*

Proof. Let $A(K)$ denote the region of attraction of K. $A(K)\backslash K$ is connected, for if $A(K)\backslash K$ were separated, then $X\backslash K$ would also be separated

because K is interior (in X) to $A(K)$. Let F be a Lyapunov function for K defined on $A(K)$. If $c > 0$ is a value of F, then $F^{-1}(c)$ is a section for the parallelizable flow π on $A(K) \backslash K$ [1, Exercise 3.14.2, p. 83]. Because $A(K) \backslash K$ is connected, $F^{-1}(c)$ must be connected [1, Lemma 2.3, p. 48].

Therefore $F^{-1}(c)$ is a compact, connected, bicollared 2-manifold imbedded in S. $F^{-1}(c)$ is either a sphere, a connected sum of tori, or a connected sum of projective planes [6, Theorem 5.1, p. 10]. Set $B = F^{-1}[0, c]$, and for each positive integer n, set $B_n = B\pi n$. Then $B_n \subset B_{n-1}$, B_i is homeomorphic to B_j for all positive integers i, j, and $K = \bigcap_{n=1}^{\infty} B_n$. By the continuity of Čech homology, $H_1(K) \approx H_1(B)$. If $F^{-1}(c) = \partial B$ were a connected sum of tori, then $H_1(B)$ would be nontrivial. If $F^{-1}(c)$ were a connected sum of projective planes, then it could not be a compact subset of X [6, p. 51]. Therefore $F^{-1}(c)$ is a 2-sphere. **

Theorem 4. *Under the hypothesis of Theorem 3, the following hold:*

(1) *$A(K)$, the region of attraction for K, is homeomorphic to an open 3-cell.*

(2) *K contains a rest point.*

(3) *There is a dynamical system $\bar{\pi}$ on X with a rest point p, such that a homeomorphism exists from $X \backslash K$ onto $X \backslash \{p\}$ which takes sensed orbits of π onto sensed orbits of $\bar{\pi}$.*

Proof. To prove (1), let c be such that $F^{-1}(c)$ is a bicollared 2-sphere, so that $B \equiv F^{-1}[0, c)$ is an open 3-cell [2a]. Then, as is easily seen, $A(K) = \bigcup_{n=1}^{\infty} B\pi(-n)$, $B\pi(-n) \subset B\pi(-n-1)$, so $A(K)$ is a monotone union of open 3-cells, hence an open 3-cell [2b].

To prove (2), we note that $F^{-1}[0, c]$ is a closed 3-cell which is invariant under the flow. Hence it must contain a rest point [1, p. 82] which is clearly in K.

Finally, we observe that K is cellular, i.e. $K = \bigcap_{n=1}^{\infty} C_n$, where $C_n \subset \text{int } C_{n-1}$ and each C_n is a 3-cell. To see this, just set $C_n = B\pi n$. Therefore, there is a continuous function $h: X \twoheadrightarrow X$ which takes K to the point p and such that $h: X \backslash K \twoheadrightarrow X \backslash \{p\}$ is a homeomorphism [7].

We define a dynamical system $(X, R, \bar{\pi})$ by

$$\bar{\pi}(x, t) = \begin{cases} p, & \text{if } x = p \\ h \circ \pi(h^{-1}(x), t), & \text{if } x \neq p \end{cases}.$$

It is easy to check that $\bar{\pi}$ satisfies the conditions of (3). **

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Theorem 5. *If $K \subset R^n$ is cellular and is asymptotically stable under* (R^n, R, π), *then:*

(1) *$A(K)$ is homeomorphic to an open n-cell.*

(2) *K contains a rest point.*

(3) *There is a dynamical system $\bar{\pi}$ on R^n with rest point p, such that a homeomorphism exists from $R^n \backslash K$ onto $R^n \backslash \{p\}$ which takes sensed orbits of π onto sensed orbits of $\bar{\pi}$.*

Proof. Because K is cellular, there is an open n-cell B, with $K \subset B \subset A(K)$. We define, for each positive integer n, $B_n = B\pi(-n)$. Then $B_n \subset$ int B_{n+1} and $A(K) = \bigcup_{n=1}^{\infty} B_n$. Therefore $A(K)$ is homeomorphic to an open n-cell [2b].

The arguments for (2) and (3) are similar to those given for Theorem 4. **

Theorem 6. *If $X = S^3$ or R^3, and (X, R, π) is a dynamical system for which K is an asymptotically stable continuum with $H_1(K) \approx Z$, then the level surfaces of a Lyapunov function for K are tori, bicollared in X.*

Proof. The proof of Theorem 3 can be slightly modified to yield this result. **

An imbedding $f: P \to Q$ of a polyhedron P into a polyhedron Q is tame if there is a homeomorphism $h: Q \longrightarrow Q$ such that $h \circ f$ is a piecewise linear imbedding. For example, a simple closed curve in R^3 or S^3 is tame if there is a space homeomorphism which takes the simple closed curve onto a polyhedral simple closed curve.

Corollary. *Suppose, in Theorem 6, K is a nontrivial periodic orbit. Then, as a simple closed curve, K is tamely imbedded in X.*

Proof. We say that K has the concentric enclosure property if $K = \bigcap_{n=1}^{\infty} T_n$, where each T_n is a solid torus, with $T_n \subset T_{n-1}$ and cl$(T_{n-1} \backslash T_n)$ is homeomorphic to the product of a torus and an interval. If we set $T = F^{-1}[0, c]$ and $T_n = T\pi n$, then $K = \bigcap_{n=1}^{\infty} T_n$.

It is not hard to show that cl$(T_n \backslash T_{n+1})$ is homeomorhpic to a torus crossed with an interval. Let $M = F^{-1}(c)$; M is a torus and M is the boundary of T. Furthermore $M\pi m$ is the boundary of $T\pi m$, so $M\pi[n-1, n] \approx T_{n-1} \backslash T_n$ so cl$[T_{n-1} \backslash T_n] \approx M \times [n-1, n]$.

Therefore K has the concentric enclosure property. Additionally, at each point p of K, there is a local section S which can be chosen to be a 2-cell, and which meets K only in p. Under these conditions, K is tamely imbedded in X [4]. **

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