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ORDERS IN SEPARABLE ALGEBRAS

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ABSTRACT. The module P^*/mP^*, where P is an order in a separable algebra over the quotient field of an integrally closed, quasi-local domain, is studied. It is shown that if the domain is complete, P^*/mP^* contains one element from each isomorphism class of irreducible P modules. Also, in general, if the global dimension of P is finite, then it equals the homological dimension of P^*/mP^*.

We are concerned with the following situation. R is an integrally closed, quasi-local domain, that is, R has a unique maximal ideal, m. K is the quotient field of R and P is an R order, projective, or what is the same thing here, free as an R module, in a separable algebra, E, over K. We will let N denote the Jacobson radical of P. We will study the module P^*/mP^* defined as follows:

$$P^* = \mathrm{Hom}_R(P, R) \quad \text{and} \quad P^*/mP^* = R/m \otimes_R P^*.$$

Now as a matter of fact $P^*/mP^* \cong (P/mP)^* = \mathrm{Hom}_{R/m}(P/mP, R/m)$. To see this consider the commutative diagram of natural maps:

$$\begin{array}{ccc} P & \xrightarrow{f} & P/mP \\ \uparrow & & \uparrow \\ R & \longrightarrow & R/m \end{array}$$

A map $P/mP \to R/m$ yields, by composition with f, a map $P \to R/m$, which lifts to a map $P \to R$, since P is a projective R module. Thus one obtains a relation $(P/mP)^* \to P^*$, and, by composition with $P^* \to P^*/mP^*$, one has a relation $(P/mP)^* \to P^*/mP^*$. It is easily checked that this relation is an isomorphism of P modules.

In our situation P^* has a special form. Since E is separable, trace(xy) (= tr(xy)), for x and y in E, is a nondegenerate, symmetric bilinear form. Any K-homomorphism, f, from E to K is uniquely represented by an element of E, say x, via $f(y) = \mathrm{tr}(xy)$, for all y in E. Given an R-homomorphism, g, from P to R we have

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$$E = K \otimes_R P \xrightarrow{1 \otimes g} K \otimes_R R = K$$

so that $1 \otimes g$ is represented by say, x, in E. If y is in P then $(1 \otimes g) \cdot (1 \otimes y) = g(y) = \mathrm{tr}(xy)$ is in R, since the elements of P are integral over R. Thus $P^* = \{x \in E \mid \mathrm{tr}(xP) \subset R\}$. In particular, $P \subset P^*$.

We are now prepared to begin establishing our first theorem about the irreducible submodules of P^*/mP^*. First we must prove two lemmas.

Lemma 1. *If x is in N, the radical of P, then* $\mathrm{tr}(x)$ *is in m.*

Proof. N/mP is the radical of P/mP, an algebra over R/m, and is, therefore, nilpotent. To compute the trace of an element x in P one takes the matrix of x in the regular representation of E and computes its trace. Now one may take as a basis of E a basis of P over R. Thus we may assume that the matrix of x has entries in R. From this it is clear that to compute the trace of an element of P/mP as an algebra over R/m we need only take a preimage of the element in P, compute its trace, and take the image of the trace in R/m. Since the elements of N/mP must have trace zero, we have shown that the elements of N must have trace in m.

Lemma 2. $mP^* = \{x \in E \mid \mathrm{tr}(xP) \subset m\}$.

Proof. Since tr is a linear map it is certainly the case that $mP^* \subset \{x \in E \mid \mathrm{tr}(xP) \subset m\}$. Suppose on the other hand, that $\mathrm{tr}(xP) \subset m$. Let $x_1, \ldots, x_n$ be a basis of P over R. P^* is also free over R on a basis $y_1, \ldots, y_n$ such that $\mathrm{tr}(y_i x_j) = \delta_{ij}$, the Kronecker delta. We have that $x = \sum_{i=1}^n r_i y_i$, $r_i \in R$, and $\mathrm{tr}(xP) = \sum_{i=1}^n r_i R \subset m$. Thus each $r_i \in m$ and $x \in mP^*$.

Theorem 1. *If R is complete (e.g. if R is a complete valuation ring or a complete local ring) then P^*/mP^* contains a representative of each isomorphism class of irreducible P modules (left and right).*

Proof. Since N is the radical of P, $P/N \cong M_{n_1}(D_1) + \cdots + M_{n_r}(D_r)$ (ring sum) with D_i a division algebra over R/m. We can write the identity of P/N as $1 = f_1 + \cdots + f_r$ where f_i is the identity of $M_{n_i}(D_i)$, $f_i f_j = 0$ if $i \neq j$, $f_i^2 = f_i$, and $f_i(P/N)f_j = 0$ if $i \neq j$.

Each isomorphism class of irreducible P modules is the unique isomorphism class of irreducibles of one of the $M_{n_i}(D_i)$, so that there are r distinct such classes. If M is an irreducible right (left) P module then it belongs to $M_{n_i}(D_i)$ if and only if $Mf_i = M$ and $Mf_j = 0$ if $j \neq i$ ($f_i M = M$ and $f_j M = 0$ if $j \neq i$).

Since R is complete and P is a finitely generated R module, we may lift the f_i respectively to e_i in P satisfying $1 = e_1 + \cdots + e_r$, $e_iPe_j \subset N$ if $i \neq j$, $e_ie_j = \delta_{ij}e_j$, $i = 1, \ldots, r$. Also, the irreducible P modules satisfy the same criteria with respect to the e_i's that they satisfy with respect to the f_i's.

Let P' denote the image of P in P^*/mP^* under the natural map $P \subset P^* \to P^*/mP^*$. We claim that $e_iP'e_i$ is a P submodule of P^*/mP^* (both left and right) for $i = 1, \ldots, r$. To prove this it is sufficient to show that e_iPe_iP is contained in $e_iPe_i + mP^*$ (we shall omit the almost identical demonstration for Pe_iPe_i). Well, since $1 = e_1 + \cdots + e_r$, we have that $P = \Sigma^r_{i,j=1} e_iPe_j$ (sum direct) so that $e_iPe_iP = \Sigma^r_{k=1} e_iPe_iPe_k$, since the e_i's are orthogonal. Certainly, $e_iPe_iPe_i$ is contained in e_iPe_i, so that we may conclude the proof of this claim if we show that $e_iPe_iPe_k$ is contained in mP^*, for $k \neq i$. Now e_iPe_k is contained in N so that $e_iPe_iPe_kP$ is too. Consequently, every element of $e_iPe_iPe_kP$ has trace in m, which, by Lemma 2, suffices.

Because $e_iP'e_i$ is an R/m module, it satisfies the descending chain condition on submodules, so that it must contain an irreducible P module, say X_i. Now $X_ie_i = X_i$ since X_i is contained in $e_iP'e_i$. If $j \neq i$ then $X_ie_j = X_ie_ie_j = 0$. Thus, by previous remarks, $X_1, \ldots, X_r$ is a complete set of irreducible P modules, and the proof of the theorem is concluded.

We shall continue in order to derive information about the homological dimension of P^*/mP^*. If S is a ring we shall denote by GD(S) the global dimension of S, dispensing with the adjectives left or right because it will either be clear or immaterial which is meant. We shall also use $d_S(M)$ to denote the projective dimension of an S module M. First we shall require some lemmas.

Lemma 3. *If S is a semiperfect ring and N is the radical of S then* GD(S) $= d_S(S/N)$.

Proof. Silver [2, Corollary 4.6].

Lemma 4. *If R is complete then* GD(P) *is the supremum of the $d_P(M)$ as M runs over the irreducible P modules.*

Proof. Lemma 3 tells us that to compute GD(P) we need only check the $d_P(P/N)$, since if R is complete, P is semiperfect. (For we can lift idempotents modulo N and P/N is semisimple.) However, P/N is a direct sum of irreducible P modules, among which appear at least one member from each isomorphism class of irreducible P modules.

Now we need to check the projective dimension of a module which contains a module of maximum projective dimension. First recall that given an exact sequence of modules $0 \to B \to A \to C \to 0$, if the projective dimensions of any two are finite, so is the third, and the dimension of A is the maximum except possibly in the case that the dimension of C is one more than the dimension of B. We prove

Lemma 5. *Let S be any ring with finite global dimension, n. If A is an S module which contains a module, B, of dimension n then the dimension of A is n.*

Proof. We have an exact sequence $0 \to B \to A \to A/B \to 0$. If $d_S(A)$ is less than n then the $d_S(A/B) = n + 1$, which is impossible.

Thus, if R is complete, Theorem 1 and Lemmas 4 and 5 allow us to conclude that if the global dimension of P is finite, it equals $d_P(P^*/mP^*)$. In the matter of projective dimension, completeness makes no difference as long as R is Noetherian, that is, local in our context. We review the facts here.

Lemma 6. *Let R be a commutative ring, L a left Noetherian R-algebra, and G an R-algebra which is a flat R module. If A is a finitely generated left L module and B is any left L module then*

$$G \otimes \operatorname{Ext}^n_L(A, B) = \operatorname{Ext}^n_{G\otimes L}(G \otimes A, G \otimes B).$$

Proof. Auslander and Goldmann [1, Lemma 2.4].

If A is a P module then A has projective dimension n if and only if $\operatorname{Ext}^{n+1}_P(A, B) = 0$ for all P modules B and there is some B with $\operatorname{Ext}^n_P(A, B) = 0$. In fact, all this is true if B is restricted to be a finitely generated module. But, in this case $\operatorname{Ext}^i_P(A, B)$ is a finitely generated R module, so that its completion is just $\hat{R} \otimes_R \operatorname{Ext}^i_P(A, B)$, where $\hat{R}$ is the completion of R. However, the completion of a module is zero if and only if the module is zero. The above plus Lemma 6 shows

Theorem 2. *If R is local and $\mathrm{GD}(P)$ is finite then $\mathrm{GD}(P) = d_P(P^*/mP^*)$.*

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