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AN INVARIANT SUBSPACE THEOREM

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ABSTRACT. If $AY - YA$ has rank one for some compact Y, then A has a nontrivial invariant subspace.

Let $\mathfrak{X}$ be a complex Banach space and $\mathfrak{B}(\mathfrak{X})$ the set of bounded linear operators on $\mathfrak{X}$. Lomonosov has proved that if A and Y belong to $\mathfrak{B}(\mathfrak{X})$ with Y compact and $AY - YA = 0$, then A has an invariant subspace [1]. We have obtained an extension of this result.

Theorem. *If $AY - YA$ has rank one for some compact Y, then A has a nontrivial invariant subspace.*

This result contrasts with the fact that if A does not commute with a trace class operator then $\{AY - YA | Y \text{ is compact}\}$ is uniformly dense in the compacts [4].

The proof of the theorem requires a lemma which may be attributed to David Luenberger although it is incorrectly stated in his paper [2]. The author is grateful to J. P. Williams who pointed out the correct version of the lemma and counterexamples to Luenberger's stronger version and partial converse.

Lemma (Luenberger). *Suppose*

$$TA - BT = C \tag{1}$$

has rank one for T, A, and B in $\mathfrak{B}(\mathfrak{X})$. If the largest A-invariant subspace of $\mathfrak{X}$ contained in the kernel of C is $\{0\}$, and the smallest B-invariant subspace of $\mathfrak{X}$ containing the range of C is $\mathfrak{X}$, then either T is one-to-one or T has dense range.

Proof. Assume that T has nontrivial kernel and nondense range in order to contradict the hypothesis.

Choose $x \neq 0$ in the kernel of T. Then $TAx = Cx$, which implies either $Cx \neq 0$ or $Ax \in \ker T$. The second alternative together with (1) yield

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$TA^2x = CAx$, hence $CAx \neq 0$ or $A^2x \in \ker T$. We may repeat this argument indefinitely. Because the span of $\{A^n x\}$ cannot be an invariant subspace for A contained in the kernel of C, there exists $y \in \ker T$ with $Cy \neq 0$.

Since $Cy = TAy$ we conclude that the kernel of A^*T^* annihilates the (one-dimensional) range of C, or $\ker A^*T^* \subset \ker C^*$. Then $A^*T^* - T^*B^* = C^*$ implies that $\ker A^*T^* \subset \ker T^*B^*$. For $x^* \in \ker A^*T^*$ we have $(A^*T^*)B^*x^* = A^*(T^*B^*)x^* = 0$, so $\ker A^*T^*$ is a B^*-invariant subspace contained in the nullspace of C^*. It follows that the closure of the range of TA is a B-invariant subspace containing the range of C, completing the proof of the lemma.

Begin the proof of the theorem by assuming that A has no invariant subspace, Y is compact, and $C = AY - YA$ has rank one. By one version of Lomonosov's theorem [3], there exists an operator B commuting with A such that $BYg = g$ for some nonzero g in $\mathfrak{X}$. Then

$$BC = B(AY - YA) = A(BY) - (BY)A = A(BY - I) - (BY - I)A$$

has rank one (B has trivial kernel since it commutes with A). Yet $BY - I$ has nontrivial kernel and by the Fredholm alternative it has nondense range. This conclusion is contrary to the lemma.

Note added in proof. Perhaps it should be remarked that if Y has rank one then the rank of $AY - YA$ is no greater than two. Thus the task of replacing "rank one" by "rank two" in the hypothesis is equivalent to solving the invariant subspace problem.

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