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A COHOMOLOGICAL CHARACTERIZATION OF PREIMAGES OF NONPLANAR, CIRCLE-LIKE CONTINUA

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ABSTRACT. Let G be an additively written Abelian group, and let $P = \{p_1, p_2, p_3, \cdots\}$ be a sequence of positive integers. An element g in G is said to have infinite P-height if (1) $g \neq 0$, (2) each $p_i > 1$, and (3) for each positive integer n, there is an element h in G such that $(p_1 p_2 \cdots p_n)h = g$. The purpose of this paper is to prove the following

Theorem. *If X is a continuum, then the following are equivalent:*

(1) *$H^1(X)$ contains an element of infinite P-height, for some sequence P of positive integers;*

(2) *X can be mapped onto a solenoid;*

(3) *X can be mapped onto a nonplanar, circle-like continuum.*

Here $H^1(X)$ is Alexander-Čech cohomology with integral coefficients.

1. **Introduction.** M. K. Fort, Jr. [3] first investigated the problem of mapping continua onto nonplanar, circle-like continua. Fort showed that no plane continuum can be mapped onto the dyadic solenoid. Tom Ingram [4] extended this result to prove that no plane continuum can be mapped onto any nonplanar, circle-like continuum. The author [9] has shown that (1) there exists an uncountable collection of nonplanar, circle-like continua (in fact, hereditarily indecomposable) such that no surjection exists from one onto another, and that (2) there exists a nonplanar, circle-like continuum that can be mapped onto any circle-like continuum. Finally, George Henderson [5] has shown that if the continuum X can be represented as the inverse limit of simply connected polyhedra, then X cannot be mapped onto a nonplanar, circle-like continuum. Henderson raises formally the problem [5, Question 1 and Problem 742], implicit in the above works, of characterizing

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the class of continua that cannot be mapped onto a nonplanar, circle-like continuum. The main theorem of this paper solves this problem; moreover, we are able to obtain the theorems of Fort, Ingram, and Henderson as easy corollaries.

The statements of the definition of infinite P-height and of the main theorem are given in the abstract. Notice that the definition of infinite P-height is a generalization of infinite height with respect to the prime p. Hence we have the following corollary to the theorem:

Corollary. *If some (nontrivial) element of $H^1(X)$ has infinite height, then X can be mapped onto a solenoid.*

2. **Preliminaries and notation.** A continuum is a compact, connected, nonvoid Hausdorff space. (Y, f) denotes the inverse limit sequence with factor spaces Y_n and bonding maps $f_m^n: Y_n \to Y_m$ $(n > m > 1)$. Let Y_∞ denote the inverse limit space of (Y, f). If each factor space $Y_n = \{z: |z| = 1\}$ (the unit circle in the complex plane) and each bonding map is a surjection, then the limit space Y_∞ is called a circle-like continuum. Furthermore, if $P = (p_1, p_2, p_3, \cdots)$ is a sequence of integers, if $p_i > 1$ for all i, and if the bonding map $f_n^{n+1}: Y_{n+1} \to Y_n$ is defined by $f_n^{n+1}(y) = y^{p_n}$, for each n, then the circle-like continuum Y_∞ is called the P-adic solenoid. If $p_n = 2$, for each n, then Y_∞ is called the dyadic solenoid.

We know from [8, Lemma 10] that without loss of generality a circle-like continuum Y_∞ satisfies one of the following conditions:

(1) $\deg f_n^{n+1} = 0$, for all n,

(2) $\deg f_n^{n+1} = 1$, for all n,

(3) $\deg f_n^{n+1} > 1$, for all n.

In the first case, $H^1(Y_\infty) \cong 0$ and Y_∞ is also arc-like and hence embeddable in the plane [2]. In the second case $H^1(Y_\infty) \cong Z$, and again Y_∞ can be embedded in the plane, this time as a separating continuum. The circle-like continua that cannot be embedded in the plane are precisely those that satisfy the third condition [1].

The Bruschlinsky group, $[X, S^1]$, is the group of all homotopy classes of maps of X into S^1. $[X, S^1]$ inherits its group structure from the group structure of S^1 and for a continuum X, $[X, S^1]$ is isomorphic to $H^1(X)$, the first Alexander-Čech cohomology group with integral coefficients. We will adopt the usual custom and denote the group operation in $[X, S^1]$ multiplicatively; this makes condition (3) in the definition of infinite P-height an exponential condition.

We will need the following lemma, which is due to M. K. Fort, Jr. [3].

Fort's lemma. *Let (E, B, p) be a locally trivial fiber space with totally disconnected fibers. If f is a mapping of a connected space A into E such that pf is homotopic to a constant, then $f(A)$ is contained in a single arc component of E.*

We refer the reader to Spanier [10] for the undefined terms of this paper.

3. **Construction of mappings.** In this section we prove the main theorem and derive its corollaries. The equivalence of (2) and (3) was first shown by Ingram [4]; our proof, using fibrations, is simpler.

Theorem. *If X is a continuum, then the following are equivalent:*

(1) *There is a sequence P of integers such that some element of $H^1(X)$ has infinite P-height;*

(2) *X can be mapped onto a solenoid;*

(3) *X can be mapped onto a nonplanar, circle-like continuum.*

Proof. (1)$\Rightarrow$(2). Since $H^1(X) \cong [X, S^1]$, there exist a nontrivial homotopy class α_1 in $[X, S^1]$ and a sequence $P = \{p_1, p_2, p_3, \cdots\}$ of positive integers such that α_1 has infinite P-height. We will show that the solenoid $Y_\infty = \varprojlim \{Y, f\}$, where $f_n^{n+1}\colon Y_{n+1} \to Y_n$ is defined by $f_n^{n+1}(y) = y^{p_n}$, is a continuous image of X. The following diagram illustrates the construction:

$$\begin{tikzcd}
 & & & X \arrow[llld, "h_1"'] \arrow[lld, "h_2"] \arrow[rrd, "h_\infty"] & & \\
Y_1 & Y_2 \arrow[l] & Y_3 \arrow[l] & \cdots \arrow[l] & : & Y_\infty
\end{tikzcd}$$

Let $h_1\colon X \to Y_1$ be a representative of α_1. The map h_1 is essential and hence surjective. Suppose that the maps $h_i\colon X \to Y_i$ have been defined for $i \leq n$ satisfying $f_{i-1}^i \circ h_i = h_{i-1}$ and h_i is essential (hence surjective). There exists $\alpha_{n+1} \in [X, S^1]$ satisfying $\alpha_{n+1}^k = \alpha_1$, where $k = p_1 p_2 \cdots p_n$. If g_{n+1} is a representative of α_{n+1}, then $f_1^{n+1} \circ g_{n+1} \sim h_1$. Since $[X, S^1]$ is torsion-free, $f_n^{n+1} \circ g_{n+1} \sim h_n$, and since f_n^{n+1} is a fibration, h_n lifts to a map $h_{n+1}\colon X \to Y_{n+1}$. Since h_n is essential, h_{n+1} is essential and hence surjective. By induction, therefore, there exists a sequence $\{h_i\colon X \to Y_i\}$ of surjections satisfying $h_{i-1} = f_{i-1}^i \circ h_i$. The sequence induces a surjection

$h_\infty: X \to Y_\infty$ defined by $h_\infty(x) = (f_1(x), f_2(x), \cdots)$.

(2) $\Rightarrow$ (1). Let $h: X \to Y_\infty$ be a surjection of X onto the solenoid Y_∞. We regard Y_∞ as $\varprojlim \{Y, f\}$, where $f_n^{n+1}: Y_{n+1} \to Y_n$ is given by $f_n^{n+1}(y) = y^{p_n}$.

Let $P = (p_1, p_2, \cdots)$, and let $f_n: Y_\infty \to Y_n$ be the projection map of Y_∞ onto Y_n. We claim that $f_1 \circ h: X \to S^1$ is a representative of a homotopy class α in $[X, S^1]$ of infinite P-height. The homotopy classes required in part (3) of the definition of infinite P-height are the homotopy classes of the maps $f_n \circ h: X \to S^1$, $n = 2, 3, \cdots$. Finally $f_1 \circ h$ is essential, by Fort's lemma, since f_1 is a fibration with fibers homeomorphic to the Cantor set, so α is nontrivial.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (2). Let $X_\infty = \varprojlim (X, g)$ be a nonplanar, circle-like continuum. We may assume that $X_n \cong S^1$ and $\deg g_n^{n+1} > 1$, for each n. Define $f_n^{n+1}: S^1 \to S^1$ by $f_n^{n+1}(y) = y^{k_n}$, where $k_n = \deg g_n^{n+1}$. Let $Y_\infty = \varprojlim (Y, f)$. We will show the solenoid Y_∞ is a continuous image of X_∞ by defining a sequence of maps $h_i: X_i \to Y_i$ of X_i onto Y_i satisfying $h_i g_i^{i+1} = f_i^{i+1} h_{i+1}$ and, hence, inducing a surjection $h_\infty: X_\infty \to Y_\infty$. Define $h_1: X_1 \to Y_1$ to be the identity map. Having defined $h_i: X_i \to Y_i$ for $i \leq n$ satisfying $\deg h_i = 1$ and $h_i g_i^{i+1} = f_i^{i+1} h_{i+1}$, define $h_{n+1}: X_{n+1} \to Y_{n+1}$ as follows: Since $h_n \circ g_n^{n+1}$ and f_n^{n+1} both have degree k_n, they are homotopic and thus h_n lifts to h_{n+1} satisfying $\deg h_{n+1} = 1$. The induction and the proof of the theorem are complete.

Corollary 1. *X can be mapped onto the dyadic solenoid if and only if $H^1(X)$ has infinite height with respect to 2.*

Corollary 2 (Fort-Ingram). *No plane continuum can be mapped onto a nonplanar, circle-like continuum.*

Proof. If X is a plane continuum, then $H^1(X)$ is isomorphic to a countable direct sum of copies of the integers by Alexander duality. No element of such a group can have infinite P-height.

Corollary 3. *If X is the inverse limit of acyclic continua, then there exists no surjection of X onto a nonplanar circle-like continuum.*

Proof. X is acyclic, by the continuity theorem.

Our final corollary is an application of the Theorem to shape theory. The definitions of any unfamiliar terms may be found in [6] and [7].

Corollary 4. *No movable continuum can be mapped onto a solenoid.*

Proof. Let X be a movable continuum. Keesling [6] has shown that if X is movable, then $H^1(X)$ has Property L. Let G be the character group of the discrete group $H^1(X)$. Then G is a compact, connected, Abelian group. Furthermore G is locally connected, since $H^1(G) \cong H^1(X)$. Hence G cannot be mapped onto a solenoid. Thus $H^1(G) \cong H^1(X)$ does not have infinite P-height for any sequence P. Therefore, X cannot be mapped onto a solenoid.

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