A NOTE ON A THEOREM OF GEHRING AND LEHTO

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ABSTRACT. The concept of mesh approximate differential is defined as a modification of regular approximate differential. It is shown that for open continuous real-valued maps on open sets in n-space the concepts of mesh approximate differentiability and total differentiability are equivalent, and the Gehring-Lehto theorem is obtained as a corollary by a sharpening of a known theorem on regular approximate differentials.

1. Introduction. A theorem of Gehring and Lehto [5], [9] asserts the existence almost everywhere (a.e.) of a total differential for an open continuous map \( f : S \to \mathbb{R}^n \) on an open set \( S \) in Euclidean \( n \)-space \( \mathbb{R}^n \), assuming that \( f \) has a total differential a.e. on \( S \) with respect to \( n - 1 \) variables. The present note obtains this theorem as a corollary of the equivalence of mesh approximate differentials (Definition 3) and total differentials in the class of real-valued open continuous maps on open sets in \( \mathbb{R}^n \) (Theorem 1), by applying a theorem of the author [3]. Thus any available sufficient conditions for the existence a.e. of a mesh approximate differential will yield total differentiability a.e. within the class of open continuous maps on open sets.

2. Notation, definitions, and basic lemmas. Let \( x = (x_1, \ldots, x^n) \) denote points in Euclidean \( n \)-space \( \mathbb{R}^n \), \( |x - y| = \left[ \sum_{i=1}^{n} (x^i - y^i)^2 \right]^{\frac{1}{2}} \) the distance between \( x \) and \( y \), and \( x \cdot y = x^1 y^1 + \cdots + x^n y^n \) the usual inner product.

**Definition 1.** For \( x_0 \in \mathbb{R}^n \) let \( \mathcal{H}(x_0) \) denote a family of oriented \((n-1)\)-hyperplanes (parallel to the coordinate planes) such that \( x_0 \) is a point of linear density of \( \bigcup H, H \in \mathcal{H}(x_0) \), in the direction of each coordinate axis. Then we term \( \mathcal{H}(x_0) \) a thick regular mesh of \((n-1)\)-hyperplanes at \( x_0 \).

**Definition 2.** For \( x_0 \in \mathbb{R}^n \) let \( \mathcal{C}(x_0) \) denote a family of oriented \( n \)-cubes (faces parallel to the coordinate planes) centered at \( x_0 \) such that \( x_0 \) is a point of density of \( \bigcup \text{fr } C, C \in \mathcal{C}(x_0) \). Then we term \( \mathcal{C}(x_0) \) a thick regular family of \( n \)-cubes at \( x_0 \).
Note that given $\mathcal{H}(x_0)$ there exists $\mathcal{C}(x_0)$ such that $\bigcup \mathcal{H}, H \in \mathcal{H}(x_0)$, is a subset of $\bigcup \mathcal{H}, H \in \mathcal{H}(x_0)$. To see this observe first that $x_0$ is a point of linear density of $\bigcup \mathcal{H}, H \in \mathcal{H}(x_0)$, in the direction of each coordinate axis, and then consider for each $i = 1, \ldots, n$ the intersections $A_i, B_i$ of this union of hyperplanes with the two rays from $x_0$ parallel to the $i$th coordinate axis. Finally construct the family $\mathcal{C}(x_0)$ by taking faces through points of $A'_i, B'_i$ ($i = 1, \ldots, n$), where $A'_i$ and $B'_i$ are those subsets of $A_i$ and $B_i$, respectively, which are congruent to the common intersection of say $A_1$ with suitable rotations of all the $A_i$ and $B_i$ ($i = 1, \ldots, n$) about $x_0$.

Definition 3. Let $f : S \to \mathbb{R}$ be a real-valued function on an open set $S$ of $\mathbb{R}^n$, and let $A$ be a subset of $S$. Assume that $f$ has partial derivatives $f'_i(x_0), i = 1, \ldots, n$, at $x_0 \in A$ relative to $A$, and let

$$e_A(x, x_0) = \frac{|f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)|}{|x - x_0|}, \quad x \in A, x \neq x_0,$$

where $\nabla f(x_0) = (f'_1(x_0), \ldots, f'_n(x_0))$. If $e_A(x, x_0) \to 0$ as $x \to x_0$ we say that $f$ has a total differential at $x_0$ relative to $A$. Then the expressions total differential at $x_0$, regular approximate differential at $x_0$, and mesh approximate differential at $x_0$ apply to the cases $A = S$, $A = \bigcup (\text{fr} C) \cap S$, $C \in \mathcal{C}(x_0)$, and $A = \bigcup \mathcal{H} \cap S, H \in \mathcal{H}(x_0)$, respectively. The notation $Df(x_0)h = \nabla f(x_0) \cdot h$ will apply to each of these differentials and the context of the usage will avoid any confusion.

Lemma 1. Let $x_0$ be a point of (linear) density of a measurable set $A \subset \mathbb{R}$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta, x \in A$, there correspond points $a, b \in A$ such that $a < x < b$, $|x - a| < \epsilon|x - x_0|$, and $|x - b| < \epsilon|x - x_0|$.

Proof. This is a direct consequence of the definition of point of linear density. The details are found essentially in [5, p. 7] and [9, p. 10].

Lemma 2. Let $\mathcal{H}(x_0)$ be a thick regular mesh of $(n - 1)$-hyperplanes at $x_0 \in S$ where $S$ is open set in $\mathbb{R}^n$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ there corresponds to $x$ an $n$-interval $I_x = \{y: a^i \leq y^i \leq b^i, i = 1, \ldots, n\}$ such that (1) $x \in I_x \subset S$, (2) $\text{fr} I_x \subset \bigcup \mathcal{H}, H \in \mathcal{H}(x_0)$, and (3) $|y - x| < \epsilon|x - x_0|$ for every $y \in \text{fr} I_x$.

Proof. Via Lemma 1, since $x_0$ is a point of linear density of $\bigcup \mathcal{H}, H \in \mathcal{H}(x_0)$, in the direction of each coordinate axis, for a given
\(\epsilon > 0\) there exist \(\delta_i > 0\) such that if \(|x^i - x^i_0| < \delta_i\) there correspond \(H^i_1, H^i_2 \in \mathcal{H}(x^i_0)\) such that \(H^i_1, H^i_2\) intersects the \(i\)th coordinate axis in the points \(a_i, b_i\) satisfying \(a^i < x^i < b^i, |x^i - a^i| < \epsilon |x^i - x^i_0|,\) and \(|x^i - b^i| < \epsilon |x^i - x^i_0|, i = 1, \ldots, n.\) Finally, choosing \(\delta_0\) so that

\[\{x: |x - x_0| < \delta_0\} \subset S,\]

set \(\delta = \min(\delta_0, \delta_1, \ldots, \delta_n)\) and \(I_x = \{y: a^i \leq y^i \leq b^i, i = 1, \ldots, n\}.\)

3. Main theorems.

Theorem 1. Let \(f: S \rightarrow R\) be a real-valued open continuous mapping on an open set \(S\) in \(R^n.\) If \(f\) has a mesh approximate differential at \(x_0 \in S,\) then \(f\) has a total differential at \(x_0.\)

Proof. By hypothesis there exists a mesh \(\mathcal{H}(x^i_0)\) of oriented \((n - 1)\)-hyperplanes such that \(x_0\) is a point of density of \(A = \bigcup H \cap S, H \in \mathcal{H}(x^i_0),\) and the total differential \(Df(x^i_0)\) exists at \(x^i_0\) relative to \(A.\) Let \(\epsilon > 0\) be given and choose \(\delta_1 > 0\) such that the conclusion of Lemma 2 is satisfied for \(\delta_1 = \delta.\) Suppose \(|x - x_0| < \delta_1, x \in S,\) and \(I_x\) is the \(n\)-interval available from Lemma 2. Since \(f\) is an open continuous map the function \(h: y \rightarrow f(y) - f(x^i_0) - Df(x^i_0)(x - x^i_0)\) restricted to \(y \in I_x\) assumes its maximum on the frontier of \(I_x,\) say \(h(y^*), y^* \in \text{fr} I_x,\) and \(|y^* - x| < \epsilon |x - x_0|.\) Observe that if we assume, as we now do, that \(\epsilon < 1,\) then \(|y^* - x^i_0| < 2|x - x^i_0|.\) Accordingly, we have

\[
|f(x) - f(x^i_0) - Df(x^i_0)(x - x^i_0)| \leq |f(y^*) - f(x^i_0) - Df(x^i_0)(y^* - x^i_0)| + |Df(x^i_0)(y^* - x^i_0)|
\]

\[
< \epsilon |y^* - x^i_0| + |\nabla f(x^i_0)||y^* - x| < 2 + |\nabla f(x^i_0)|\epsilon |x - x^i_0|
\]

if \(|x - x^i_0| < \min(\delta_1, \delta_2)\) where \(\delta_2\) is selected so that we have

\[
|f(z) - f(x^i_0) - Df(x^i_0)(z - x^i_0)| < \epsilon |z - x^i_0|
\]

if \(z \in A\) and \(|z - x^i_0| < \delta_2.\) Thus \(f\) has a total differential at \(x^i_0\) and the proof is complete.

A theorem of the author [3] asserts that for continuous real-valued maps on open sets in \(R^n\) the existence a.e. of a regular approximate differential (Definition 3) is a consequence of the existence a.e. of a total differential with respect to \(n - 1\) variables. Inspection of the proof reveals that "regular approximate" can be strengthened to "mesh approximate" by considering instead of a face of an \(n\)-cube the \((n - 1)\)-hyperplane containing the face. For reference we state this result formally.
Theorem 2. Given a continuous real-valued map $f : S \rightarrow \mathbb{R}$ on an open set $S$ in $\mathbb{R}^n$, suppose $f$ has a total differential a.e. with respect to $n - 1$ variables. Then $f$ has a mesh approximate differential a.e. on $S$.

In view of Theorem 1 we infer from Theorem 2 the theorem of Gehring and Lehto for real-valued maps, which when applied to the coordinate maps of a given map $f : S \rightarrow \mathbb{R}^m$ yields the following form of the theorem:

Gehring-Lehto theorem. Given a map $f : S \rightarrow \mathbb{R}^m$ on an open set $S$ in $\mathbb{R}^n$, suppose the $m$ coordinate maps are continuous open maps on $S$. Then if $f$ has a total differential a.e. on $S$ with respect to $n - 1$ variables, then $f$ has a total differential a.e. on $S$.

An example of Väisälä [9, p. 11], shows that in the Gehring-Lehto theorem for $n \geq 3$ the hypothesis of total differentiability with respect to $n - 1$ variables a.e. may not be replaced by the existence a.e. of simply partial derivatives. Accordingly since the theorem is a consequence of Theorem 1 and Theorem 2, we remark that the existence a.e. of partial derivatives for $n \geq 3$ does not imply the existence a.e. of a mesh approximate differential a.e., so that in a certain sense Theorem 2 is sharp.

REFERENCES


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