PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 1, May 1975

$\mathcal{T}$ MEASURE OF CARTESIAN PRODUCT SETS[1]

LAWRENCE R. ERNST

ABSTRACT. It is proven that there exists a subset A of Euclidean 2-space such that the 2-dimensional $\mathcal{T}$ measure of the Cartesian product of an interval of unit length and A is greater than the 1-dimensional $\mathcal{T}$ measure of A. This shows that $\mathcal{T}$ measure does not extend to Euclidean 3-space the relation that area is the product of length by length. As corollaries, new proofs of some related but previously known results are obtained.

1. **Introduction.** There are many 1-dimensional measures and 2-dimensional measures over Euclidean n-space, $\mathbf{R}^n$, which generalize the concept of length and area respectively. These measures were studied extensively by H. Federer in [4]. A natural question concerning them, first posed by J. F. Randolph in [7], is whether they extend to $\mathbf{R}^3$ the relation of area being the product of length by length. Answers to this question, positive for certain measures and negative for others, were obtained for a number of these measures and are listed in [4, 2.10.46]. However, no solution had been obtained until now for $\mathcal{T}$ measure, which was first defined in [4, 2.10.3].

The main purpose of this paper is to present a negative answer for $\mathcal{T}$ measure, which we do by constructing a subset A of $\mathbf{R}^2$ that we prove satisfies the relation $\mathcal{T}^2(\{x: 0 \leq x \leq 1\} \times A) > \mathcal{T}^1(A)$ (Theorem 3.6). As a corollary to this result we obtain that $\mathcal{T}$ measure can be replaced in this inequality by either Hausdorff measure or spherical measure (Corollary 3.7). We also deduce that the 2-dimensional Carathéodory measure of $\{x: 0 \leq x \leq 1\} \times A$ is less than its 2-dimensional $\mathcal{T}$ measure (Corollary 3.8), thus showing that these measures are distinct. The results of these corollaries were obtained previously, using different examples and methods, by A. S. Besicovitch and P. A. P. Moran [1] and G. Freilich [5] in the case of Cor-

Received by the editors January 30, 1974.
AMS (MOS) subject classifications (1970). Primary 28A35, 28A75.
Key words and phrases. 1-dimensional measures, 2-dimensional measures, Cartesian product sets, $\mathcal{T}$ measure, Hausdorff measure, spherical measure, Carathéodory measure.
[1] Research supported by a grant from the City University of New York Faculty Research Award Program, # 10163.

ollary 3.7, and by the author [2] in the case of Corollary 3.8.

2. **Preliminaries.** In general we adopt in this paper the notation and terminology of [4]. Presented in this section are additional definitions that we use.

To define the set A referred to in the introduction, first inductively define families $G_0, G_1, G_2, \ldots$ of closed circular disks contained in $\mathbf{R}^2 = \mathbf{C}$ by taking

$$G_0 = \{\mathbf{B}(0, \tfrac{1}{2})\},$$

$$G_j = \{\mathbf{B}[z + 0.99r \exp(0.02\pi k\mathrm{i}), 0.01r] : \mathbf{B}(z, r) \in G_{j-1}, k = 1, \ldots, 100\}$$

for $j \geq 1$; then let $A = \bigcap_{j=0}^{\infty} \bigcup G_j$; furthermore, abbreviate $\{x: 0 \leq x \leq 1\} \times A$ by E.

For $S \subset A$ with $\operatorname{diam} S > 0$ define

$$\zeta(S) = \sup\{j : S \subset T \text{ for some } T \in G_j\} + 1$$

and

$$\lambda(S) = G_{\zeta(S)} \cap \{T : T \cap S \neq \emptyset\}.$$

Let $S - T = \{x - y : x \in S, y \in T\}$ for $S, T \subset \mathbf{R}^n$.

Finally, for $\emptyset \neq S \subset \mathbf{R}^n$ let

$$t^1(S) = \operatorname{diam} S$$

and

$$t^2(S) = (\pi/4) \sup\{|(a_1 - b_1) \wedge (a_2 - b_2)| : a_1, b_1, a_2, b_2 \in S\}.$$

These are the gauge functions used in defining $\mathcal{T}^1$ and $\mathcal{T}^2$ respectively [4, 2.10.3].

3. **Principal results.**

3.1. **Lemma.** *If* $S \subset A$, $\operatorname{diam} S > 0$ *and* $\operatorname{card} \lambda(S) \leq 51$, *then there exist* $U, V \in \lambda(S)$ *such that*

$$\operatorname{dist}(U, V) \geq [99 \sin([\operatorname{card} \lambda(S) - 1]0.01\pi) - 1]10^{-2\zeta(S)}.$$

Proof. The conclusion follows from the observation that for some $U, V \in \lambda(S)$ the distance between the centers of U and V is at least $[99 \sin([\operatorname{card} \lambda(S) - 1]0.01\pi)10^{-2\zeta(S)}$.

3.2. **Lemma.** $\mathcal{T}^1(A) \leq 1$.

Proof. To obtain this inequality we simply note that G_j is a covering of A by circular disks of diameter 10^{-2j} and $\Sigma_{S \in G_j} t^1(S) = 1$.

3.3. **Lemma.** $\mathcal{T}^1(A) > 0$.

Proof. Consider any countable covering of A consisting of nonempty subsets of A that are open in A, and let W be a finite subcover. Since $\bigcup_{S \in W} \lambda(S)$ covers A, and $T \cap A \neq \emptyset$ for any $T \in \bigcup_{j=0}^{\infty} G_j$, it follows that

$$\sum_{S \in W} 10^{-2\zeta(S)+2} \geq \sum_{S \in W} [\operatorname{card} \lambda(S)] 10^{-2\zeta(S)} \geq 1.$$

Using this result, Lemma 3.1, and the fact that $\operatorname{card} \lambda(S) \geq 2$ for all $S \in W$, we deduce that

$$\sum_{S \in W} t^1(S) \geq \sum_{S \in W} [99 \sin(0.01\pi) - 1] 10^{-2\zeta(S)} \geq 0.99 \sin(0.01\pi) - 0.01;$$

hence $\mathcal{T}^1(A) > 0$.

3.4. **Corollary.** $0 < \mathcal{T}^2(E) < \infty$.

Proof. We combine Lemmas 3.2 and 3.3, [4, 2.10.45] and the fact that the ratios between $\mathcal{T}^m$ and $\mathcal{H}^m$ are bounded [4, 2.10.6].

3.5. **Lemma.** *If B is a closed subset of E and $\mathcal{T}^2(B) > 0$, then $t^2(B) > 1.1\mathcal{T}^2(B)/\mathcal{T}^2(E)$.*

Proof. Define $p\colon \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$, $q\colon \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$, $p(x, (y, z)) = x$, $q(x, (y, z)) = (y, z)$ for $(x, (y, z)) \in \mathbf{R} \times \mathbf{R}^2$. Abbreviate $C = q(B)$, $n = \operatorname{card} \lambda(C)$. For $S \subset \mathbf{R}^2$ let $\psi(S) = p[q^{-1}(S) \cap B]$. Choose $D \in \lambda(C)$ satisfying $\mathcal{L}^1[\psi(D)] = \inf\{\mathcal{L}^1[\psi(S)]\colon S \in \lambda(C)\}$. Further let: $M = \lambda(C) \sim \{D\}$ if n is odd and $M = \lambda(C)$ if n is even; $k = \operatorname{card} M$; $S_1, \ldots, S_k$ denote a sequence consisting of all the elements of M arranged in clockwise order; and

$$\rho = \sup\{\mathcal{L}^1[\psi(S_i)] + \mathcal{L}^1[\psi(S_{i+k/2})]\colon i = 1, \ldots, k/2\}.$$

Now clearly $\mathcal{T}^2(B)/\mathcal{T}^2(E) \leq 0.5n\rho 10^{-2\zeta(C)}$. Consequently, to obtain our conclusion it suffices to show that

$$t^2(B) > 0.55n\rho 10^{-2\zeta(C)}. \tag{1}$$

To prove (1) we first choose $j \in \{1, \ldots, k/2\}$ for which $\mathcal{L}^1[\psi(S_j)] + \mathcal{L}^1[\psi(S_{j+k/2})] = \rho$, and let $d = \operatorname{dist}(S_j, S_{j+k/2})$. We deduce that

$$d > 0.702n10^{-2\zeta(C)} \tag{2}$$

by applying Lemma 3.1 to obtain that $d \geq [99 \sin(0.005\pi k) - 1]10^{-2\zeta(C)}$ and then finding that the minimum value of $[99 \sin(0.005\pi k) - 1]/n$ for $n = 2, 3, \ldots, 100$, which occurs when $n = 3$, is greater than 0.702. We next choose $u, v \in [q^{-1}(S_j) \cap B] - [q^{-1}(S_{j+k/2}) \cap B]$ satisfying $p(u) = \sup \psi(S_j) - \inf \psi(S_{j+k/2})$, $p(v) = \inf \psi(S_j) - \sup \psi(S_{j+k/2})$, and observe that

$$p(u) - p(v) \geq \rho. \tag{3}$$

Finally, we choose $f \in \mathbf{O}^*(2, 1)$ satisfying $f(w) \geq d$ for all $w \in S_j - S_{j+k/2}$, and use (2) and (3) to conclude that

$$t^2(B) \geq (\pi/4)|u \wedge v| \geq (\pi/4)(p(u)f[q(v)] - p(v)f[q(u)])$$

$$\geq (\pi/4)d[p(u) - p(v)] > 0.55n\rho 10^{-2\zeta(C)}.$$

3.6. **Theorem.** $\mathcal{T}^2(E) > 1 \geq \mathcal{T}^1(A)$.

Proof. To establish that $\mathcal{T}^2(E) > 1$ we consider any countable covering W of E consisting of nonempty closed subsets of E and apply Lemma 3.5 and Corollary 3.4 to obtain that

$$\sum_{S \in W} t^2(S) > 1.1 \sum_{S \in W} \mathcal{T}^2(S)/\mathcal{T}^2(E) \geq 1.1.$$

3.7. **Corollary.** $\mathcal{H}^2(E) > \mathcal{H}^1(A)$ *and* $\mathcal{S}^2(E) > \mathcal{S}^1(A)$.

Proof. The proof of Lemma 3.2 establishes that $\mathcal{H}^1(A) \leq 1$ and $\mathcal{S}^1(A) \leq 1$, while Theorem 3.6 and the fact that $\mathcal{S}^2(S) \geq \mathcal{H}^2(S) \geq \mathcal{T}^2(S)$ for all $S \subset \mathbf{R} \times \mathbf{R}^2$ [4, 2.10.6] yield the inequalities $\mathcal{H}^2(E) > 1$ and $\mathcal{S}^2(E) > 1$.

3.8. **Corollary.** $\mathcal{C}^2(E) < \mathcal{T}^2(E)$.

Proof. It follows from Theorem 3.6 and [4, 2.10.34] that $\mathcal{C}^1(A) < \mathcal{T}^2(E)$, and from [4, 2.10.46] that $\mathcal{C}^2(E) \leq \mathcal{C}^1(A)$.

3.9. **Remark.** In [3] it was shown that $\mathcal{T}^2(E) < \mathcal{H}^2(E)$. Thus the existence of a set E for which $\mathcal{C}^2(E) < \mathcal{T}^2(E) < \mathcal{H}^2(E)$ has been established.

REFERENCES

1. A. S. Besicovitch and P. A. P. Moran, *The measure of product and cylinder sets*, J. London Math. Soc. 20 (1945), 110–120. MR 8, 18.

2. L. R. Ernst, *A proof that $\mathcal{C}^2$ and $\mathcal{T}^2$ are distinct measures*, Trans. Amer. Math. Soc. 173 (1972), 501–508. MR 46 #9266.

3. L. R. Ernst, *A proof that $\mathcal{H}^2$ and $\mathcal{T}^2$ are distinct measures*, Trans. Amer. Math. Soc. 191 (1974), 363–372.

4. H. Federer, *Geometric measure theory*, Die Grundlehren der math. Wissenschaften, Band 153, Springer-Verlag, New York, 1969. MR 41 #1976.

5. G. Freilich, *On the measure of Cartesian product sets*, Trans. Amer. Math. Soc. 69 (1950), 232–275. MR 12, 324.

6. ———, *Carathéodory measure of cylinders*, Trans. Amer. Math. Soc. 114 (1965), 384–400. MR 30 #4892.

7. J. F. Randolph, *On generalizations of length and area*, Bull. Amer. Math. Soc. 42 (1936), 268–274.

DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE (CUNY), FLUSHING, NEW YORK 11367