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NEW CRITERIA FOR UNIVALENT FUNCTIONS

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ABSTRACT. The classes K_n of functions $f(z)$ regular in the unit disc $\mathfrak{U}$ with $f(0) = 0, f'(0) = 1$ satisfying

$$\mathrm{Re}\left[(z^n f)^{(n+1)}/(z^{n-1}f)^{(n)}\right] > (n+1)/2$$

in $\mathfrak{U}$ are considered and $K_{n+1} \subset K_n$, $n = 0, 1, \cdots$, is proved. Since K_0 is the class of functions starlike of order ½ all functions in K_n are univalent. Some coefficient estimates are given and special elements of K_n are determined.

1. **Introduction.** Let A denote the family of functions $f(z)$ regular in the unit disc $\mathfrak{U} = \{|z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$. In this paper we shall show that a function $f \in A$ which satisfies one of the conditions

$$\mathrm{Re}\,\frac{(z^n f)^{(n+1)}}{(z^{n-1}f)^{(n)}} > \frac{n+1}{2}, \qquad z \in \mathfrak{U}, \tag{1.1}$$

$n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, is univalent in $\mathfrak{U}$. More precisely it is proved that for the classes K_n of functions $f \in A$ satisfying (1.1)

$$K_{n+1} \subset K_n, \qquad n \in \mathbf{N}_0, \tag{1.2}$$

holds. Since K_0 equals $S^*_{1/2}$ (the class of functions starlike of order ½) the univalence of the members in K_n will be a consequence of (1.2). Furthermore we have $K_1 = K$, where K consists of the functions $f \in A$ that map U univalently onto convex domains. Therefore (1.2) is an extension of Strohhäcker's result $\underline{K \subset S^*_{1/2}}$ [4]. Another important relation between K and $S^*_{1/2}$ ($f \in K \Leftrightarrow z\sqrt{f'(z)} \in S^*_{1/2}$) will also be extended to K_n.

Next we deduce estimates for $|a_k - \mu a_2^{k-1}|$, $k = 3, 4, \cdots$, for $f(z) = \sum_{j=1}^{\infty} a_j z^j \in K_n$, which enable us to prove

$$\bigcap_{n \in \mathbf{N}_0} K_n = \left\{ \frac{z}{1 - xz} \,\middle|\, |x| \le 1 \right\}. \tag{1.3}$$

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Examples for elements in K_n are the functions

$$h_\gamma(z) = \sum_{j=1}^{\infty} \frac{\gamma+1}{\gamma+j} z^j, \qquad \operatorname{Re} \gamma \geq \frac{n-1}{2},$$

which also have an interesting convolution property. This extends a result due to G. Pólya and I. J. Schoenberg [2] (the case $\operatorname{Re} \gamma = 0$). To conclude this paper we shall mention some conjectures concerning K_n.

2. **The classes** K_n. The Hadamard product or convolution of two functions $f, g \in A$ will be denoted by $f * g$. Let

$$D^\alpha f = (z/(1-z)^{\alpha+1}) * f, \qquad \alpha \geq -1, \tag{2.1}$$

which implies

$$D^n f = z(z^{n-1} f)^{(n)}/n!, \qquad n \in \mathbf{N}_0. \tag{2.2}$$

With this notation the well-known conditions for a function $f \in A$ to be in $S^*_{1/2}$ or K can be written as

$$\begin{aligned} f \in S^*_{1/2} &\Leftrightarrow \operatorname{Re}(D^1 f/D^0 f) > ½, \qquad z \in \mathfrak{U}, \\ f \in K &\Leftrightarrow \operatorname{Re}(D^2 f/D^1 f) > ½, \qquad z \in \mathfrak{U}. \end{aligned} \tag{2.3}$$

(1.1) and (2.2) give for $f \in A$

$$f \in K_n \Leftrightarrow \operatorname{Re}(D^{n+1} f/D^n f) > ½, \qquad z \in \mathfrak{U}, \tag{2.4}$$

so that the definition of K_n is a natural extension of (2.3). In the notation (2.4) also a class K_{-1} can be defined: as the set of functions $f \in A$ with $\operatorname{Re} f(z)/z > ½$, $z \in \mathfrak{U}$. Some of the following theorems hold also in this case.

Theorem 1. *$K_{n+1} \subset K_n$ holds for $n \in \mathbf{N}_0$.*

For the proof we need a lemma due to I. S. Jack [1].

Lemma 1. *Let w be nonconstant and regular in $|z| < R$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < R$ at z_0, we have $z_0 w'(z_0)/w(z_0) \geq 1$.*

Proof of Theorem 1. Let $f \in K_{n+1}$, $g(z) = z[(D^n f)/z]^{2/(n+1)}$ and $R = \sup\{r \mid g(z) \neq 0,\ 0 < |z| < r < 1\}$. Then $g(z)$ is single valued in $|z| < R$ and $u(z) = zg'(z)/g(z)$ is regular in this circle. The recursion formula

$$\frac{z}{(1-z)^{n+2}} = \frac{z}{(1-z)^{n+1}} * \left[\frac{n}{n+1}\frac{z}{1-z} + \frac{1}{n+1}\frac{z}{(1-z)^2}\right] \tag{2.5}$$

leads to the identities

$$\frac{D^{n+2}f}{D^{n+1}f} - \frac{1}{2} = \frac{1}{n+2}\left[\frac{1}{2} + \frac{n+1}{2}u + \frac{zu'}{u+1}\right]$$

and

$$D^{n+1}f/D^n f - 1/2 = u/2. \tag{2.6}$$

The condition

$$\operatorname{Re}\left[\frac{zu'}{u+1} + \frac{n+1}{2}u\right] > -\frac{1}{2}, \qquad |z| < R,$$

implies $\operatorname{Re} u > 0$, $|z| < R$. In fact, let $u = (1+w)/(1-w)$, so that

$$\frac{zu'}{u+1} + \frac{n+1}{2}u = \frac{zw'}{w}\,\frac{w}{1-w} + \frac{n+1}{2}\,\frac{1+w}{1-w}. \tag{2.7}$$

If $\operatorname{Re} u(z_0) = 0$ for a certain z_0, $|z_0| < R$, and $\operatorname{Re} u(z) \geq 0$, $|z| \leq |z_0|$, then $|w(z)| \leq |w(z_0)| = 1$ for $|z| \leq |z_0|$ and $w(z_0) \neq 1$. Lemma 1 and (2.7) give

$$\operatorname{Re}\left[\frac{z_0 u'(z_0)}{u(z_0)+1} + \frac{n+1}{2}u(z_0)\right] = -\frac{1}{2}\,\frac{z_0 w'(z_0)}{w(z_0)} \leq -\frac{1}{2},$$

a contradiction to the assumption. $\operatorname{Re} u(z) > 0$ for $|z| < R$ implies the univalence of $g(z)$ in that circle. Thus it is not possible that $g(z)$ vanishes on $|z| = R$, if $R < 1$. So we conclude $R = 1$ and the proof is completed.

From Theorem 1 and (2.4) it follows that $D^n f \neq 0$, $0 < |z| < 1$, for $f \in K_n$. If we put

$$g(z) = z[(D^n f)/z]^{1/(n+1)} \tag{2.8}$$

we receive $(D^{n+1}f)/(D^n f) = zg'/g$ (compare (2.6)). Since equation (2.8) is uniquely solvable for $f \in A$ we have the result that $f \in K_n$ if and only if $g \in K_0$. The following theorem reflects this observation.

Theorem 2. *Let $m, n \in \mathbf{N}_0$. Then $f \in K_n$ if and only if*

$$g = m! z^{1-m} \int_0^z \int_0^{x_m} \cdots$$

$$\int_0^{x_2} \left[\frac{1}{n!}(x_1^{n-1} f(x_1))^{(n)}\right]^{(m+1)/(n+1)} dx_1 \cdots dx_m \in K_m.$$

3. **Coefficient estimates.** Let $f(z) = \sum_{j=1}^{\infty} a_j z^j \in S^*_{1/2}$. It is well known that in this case $|a_k| \leq 1$, $k = 2, 3, \cdots$, holds and that $f_0(z) = z/(1-z)$ is an extremal function. Since $f_0 \in K_n \subset S^*_{1/2}$, $n \in \mathbf{N}_0$, it becomes clear that this estimate cannot be improved for $f \in K_n$. On the other hand the common techniques lead to

Theorem 3. *Let* $f = \sum_{j=1}^{\infty} a_j z^j \in K_n$. *Then we have the sharp estimate*

$$|a_3 - a_2^2| \leq (1 - |a_2|^2)/(n + 2). \tag{3.1}$$

For $f \in \bigcap_{n \in \mathbf{N}_0} K_n$ this result implies $a_3 = a_2^2$. This fact increases the interest in estimates of the functional $|a_k - a_2^{k-1}|$ over K_n.

Theorem 4. *Let* $f = \sum_{j=1}^{\infty} a_j z^j \in K_n$ *and*

$$\gamma(n, k) = \binom{n+1}{k-1} \Big/ \binom{n+k-1}{k-1}. \tag{3.2}$$

Then, for $\mu \leq \gamma(n, k)$, *we have the sharp estimate*

$$|a_k - \mu a_2^{k-1}| \leq 1 - \mu, \qquad k = 3, 4, \cdots. \tag{3.3}$$

Proof. Theorem 2 gives

$$f(z) = n! z^{1-n} \int_0^z \int_0^{x_n} \cdots \int_0^{x_2} [h(x_1)/x_1]^{n+1} \, dx_1 \cdots dx_n,$$

where $h(z) = \sum_{j=1}^{\infty} b_j z^j \in S_{1/2}^*$ and $a_2 = b_2$. If we put $(h(z)/z)^{n+1} = \sum_{j=0}^{\infty} c_{j+1} z^j$, $(1 - z)^{-n-1} = \sum_{j=0}^{\infty} d_{j+1} z^j$, then, with $\sigma = \mu\binom{n+k-1}{k-1}$, we receive

$$c_k - \sigma b_2^{k-1} = F(b_2, b_3, \cdots, b_k) + \left[\binom{n+1}{k-1} - \sigma\right] b_2^{k-1},$$

$$d_k - \sigma = F(1, 1, \cdots, 1) + \binom{n+1}{k-1} - \sigma = \binom{n+k-1}{k-1} - \sigma.$$

Obviously $|F(b_2, b_3, \cdots, b_k)| \leq F(1, 1, \cdots, 1)$ and with $\sigma \leq \binom{n+1}{k-1}$ and $c_k = \binom{n+k-1}{k-1} a_k$, (3.3) follows. Clearly (3.3) is sharp for $z/(1 - z)$.

Corollary 1. *Let* $f = \sum_{j=1}^{\infty} a_j z^j \in K_n$. *Then, with the notation of* (3.2), *we have*

$$|a_k - a_2^{k-1}| \leq 2(1 - \gamma(n, k)), \qquad k = 3, 4, \cdots. \tag{3.4}$$

Furthermore

$$|a_k - a_2^{k-1}| \leq \alpha(n) k^2/n + M n^{-2}, \qquad k = 3, 4, \cdots, \tag{3.5}$$

where $\alpha(n) < 5.7$ *and* $\overline{\lim}_{n \to \infty} \alpha(n) \leq 4$. M *is an absolute constant.*

Proof. (3.4) follows immediately from (3.3) and the triangular inequality. For the proof of (3.5) it is enough to show

(3.6) $$\gamma(n, k+1) \geq 1 - k^2/(\beta^2 n) + M^* n^{-2},$$

where β denotes the positive root of the equation $2\beta^2 + \beta\sqrt{n}/(n+1) = 1$. Since $\gamma \geq 0$ it suffices to prove (3.6) for $0 < k < \beta\sqrt{n}$. We use Stirling's formula

$$p! = (p/e)^p \sqrt{2\pi p}\,(1 + 1/12p + \delta(p))$$

with $\delta(p) = O(p^{-2})$ and remark that in the considered interval for k we have $\delta(n+k) = O(n^{-2})$, $\delta(n-k) = O(n^{-2})$. Now

$$\left(1 + \frac{1}{12n} + \delta(n)\right)^2 \left(1 + \frac{1}{12(n+k)} + \delta(n+k)\right)^{-1} \left(1 + \frac{1}{12(n-k)} + \delta(n-k)\right)^{-1} = 1 + O(n^{-2}),$$

and using Bernoulli's inequality we receive

$$\gamma(n, k+1) = \frac{n+1}{n+1-k}\left(\frac{n^2}{n^2-k^2}\right)^{n+1/2}\left(\frac{n-k}{n+k}\right)^k + O(n^{-2})$$

$$\geq \frac{n+1}{n+1-k}\left(1 - \frac{k}{n}\right)^{2k} + \frac{M^*}{n^2} \geq 1 + \frac{n+1}{n+1-k}\left(\frac{k}{n+1} - \frac{2k^2}{n}\right) + \frac{M^*}{n^2}$$

$$\geq 1 - 2\left(1 - \frac{\beta\sqrt{n}}{n+1}\right)^{-1}\frac{k^2}{n} + \frac{M^*}{n^2} = 1 - \frac{k^2}{\beta^2 n} + \frac{M^*}{n^2}.$$

Corollary 2. $\bigcap_{n \in \mathbf{N}_0} K_n = \{z/(1 - xz) \mid |x| \leq 1\}$.

Proof. That these functions are elements of K_n is easily seen from (2.4). The other direction follows from Theorem 1 and (3.5).

4. Special elements of K_n. In this section we consider the functions

$$h_\gamma(z) = \sum_{j=1}^{\infty} \frac{\gamma+1}{\gamma+j} z^j, \qquad \operatorname{Re}\gamma > -1.$$

In connection with their work on variation-diminishing transformations G. Pólya and I. J. Schoenberg [2] proved $h_{i\alpha} \in K$, $\alpha \in \mathbf{R}$, and $h_{i\alpha} * f \in K$ for $f \in K$. Theorem 5 extends their result.

Theorem 5. *Let $n \in \mathbf{N}_0$, $\operatorname{Re}\gamma \geq (n-1)/2$. Then $f * h_\gamma \in K_n$ for all $f \in K_n$. In particular $h_\gamma \in K_n$.*

Proof. Since the families K_n are compact it is enough to prove the theorem for $\operatorname{Re}\gamma > (n-1)/2$. The function $F = f * h_\gamma$ satisfies

$$zF' + \gamma F = (1+\gamma)f. \tag{4.1}$$

Again we put $g = z[(D^nF)/z]^{1/(n+1)}$ and $R = \sup\{r \mid g(z) \neq 0,\ 0 < |z| < r < 1\}$. We shall prove $\operatorname{Re} zg'/g > ½$, $|z| < R$, and this implies $R = 1$. Let $(D^{n+1}F)/(D^nF) = zg'/g = 1/(1 - w(z))$. Using (2.5) and (4.1) we obtain

$$\left[\frac{zw' + n + 1}{(1-w)^2} + \frac{\gamma - n}{1 - w}\right] D^nF = (1+\gamma)D^{n+1}f,$$

$$\left[\frac{n+1}{1-w} + \gamma - n\right] D^nF = (1+\gamma)D^nf,$$

and consequently

$$\frac{D^{n+1}f}{D^nf} = \frac{zw'}{(1-w)^2}\left(\frac{n+1}{1-w} + \gamma - n\right)^{-1} + \frac{1}{1-w}.$$

Again we use Lemma 1. Let $1 = |w(z_0)| \geq |w(z)|$ for $|z| \leq |z_0| < R$. Then

$$\operatorname{Re} \left.\frac{D^{n+1}f}{D^nf}\right|_{z=z_0} - \frac{1}{2} = \mu \frac{z_0w'(z_0)}{w(z_0)} \operatorname{Re}\left(\frac{n+1}{1 - w(z_0)} + \gamma - n\right)^{-1},$$

where $\mu = -w(z_0)(1 - w(z_0))^{-2} \geq ¼$. In the case $\operatorname{Re}\gamma > (n-1)/2$ this contradicts the assumption $f \in K_n$. Since $h_\gamma = (z/(1-z)) * h_\gamma$ the second part is an easy consequence of the first one.

The first part of Theorem 5 clearly can also be written in the following perhaps more suggestive form: Let $\operatorname{Re}\gamma \geq (n-1)/2$. If $f \in K_n$ then so is $z^{-\gamma}\int_0^z t^{\gamma-1}f(t)\,dt$. By (also infinite) iteration of the Hadamard product of suitable functions h_γ it is possible to construct a broad class of functions in K_n, similar to the subclass of K used by Pólya and Schoenberg in their paper mentioned above.

5. **Problems.** In this paper we treated only a few properties of the functions in K_n. The presence of higher derivatives in their definition makes the application of the usual techniques difficult. On the other hand, methods involving Hadamard products seem to be more suitable for the treatment, but they are not well developed at the moment, at least in the view of geometrical applications. The following questions arise naturally from the results of this paper.

(i) What can be said about the classes K_α, if we replace the natural number n in (2.4) by an arbitrary real number $\alpha \geq -1$. Is it perhaps true that $K_\alpha \subset K_\beta$ for $\alpha > \beta$?

(ii) Is K_α closed under the Hadamard product?

The truth of (ii) is trivial for $\alpha = -1$ and proved in [3] for $\alpha = 0, 1$. We conjecture that the answer in all cases is affirmative.

Nothing is known about the geometrical structure of the elements in K_n (beside their convexity for $n \geq 1$). The truth of the following conjecture (already proved for $n = -1, 0$) would throw some light on this problem: $f \in K_{n+1}$ if and only if for all $z_0 \in \mathfrak{U}$ we have

$$\frac{zz_0}{f(z_0)} \frac{f(z) - f(z_0)}{z - z_0} \in K_n.$$

Another interesting problem is the determination of the smallest values δ_n, such that the condition $\operatorname{Re} (D^{n+1}f/D^n f) > \delta_n$, $z \in \mathfrak{U}$, guarantees the univalence of $f \in A$. It is well known that $\delta_0 = 0, \delta_1 = \frac{1}{4}$.

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