COMPLETE DOMAINS WITH RESPECT TO THE CARATHÉODORY DISTANCE

DONG S. KIM

ABSTRACT. Concerning completeness with respect to the Carathéodory distance ($c$-completeness), the following theorems are shown. A bounded convex (in geometric sense) domain $D$ in $\mathbb{C}^n$ ($\mathbb{R}^{2n}$) is $c$-complete, so that it is boundedly holomorphic convex. To preserve $c$-completeness in complex spaces, it is sufficient to have a proper local biholomorphic mapping as follows: Let $\alpha$ be a proper spread map of a $c$-hyperbolic complex space $(\tilde{X}, \tilde{A})$ onto a $c$-hyperbolic complex space $(\tilde{X}, \tilde{A})$; then $X$ is $c$-complete if and only if $\tilde{X}$ is $c$-complete. We also show the following $D$ to be domains of bounded holomorphy: let $(X, A)$ be a Riemann domain and $D$ a domain in $X$ with $\alpha(D)$ bounded in $\mathbb{C}^n$. Let $B(D)$ separate the points of $D$. Suppose there is a compact set $K$ such that for any $x \in D$ there is an analytic automorphism $\sigma \in \text{Aut}(D)$ and a point $a \in K$ such that $\sigma(x) = a$. Then $D$ is a domain of bounded holomorphy.

Let $(X, A)$ be a complex space and $D$ a domain (open and connected) in $X$. Let $B = B(D)$ be the algebra of bounded holomorphic functions on $D$ and

$$B_1 = \{ f \in B; \sup_{x \in D} |f(x)| = \|f\|_D = 1 \}.$$

We define the Carathéodory distance $c = c_D$ as follows: For $x, y \in D$,

$$c(x, y) = \sup_{g \in B_1} \rho(g(x), g(y)),$$

where

$$\rho(z_1, z_2) = \log \frac{|z_2 - z_1| + |1 - z_1 \overline{z_2}|}{\sqrt{(1 - z_1 \overline{z_1})(1 - z_2 \overline{z_2})}},$$

where $z_1, z_2$ are in the open unit disc in $\mathbb{C}$.

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For \( g \in B_1 \) and \( x' \in D \), set
\[
f(x') = \frac{g(x') - g(x)}{g(x')g(x) - 1};
\]
then
\[
c(x, y) = \sup_{f \in B_X} \left\{ \frac{1}{2} \log \frac{1 + |f(y)|}{1 - |f(y)|} \right\},
\]
where \( B_X = \{ f \in B_1; f(x) = 0 \} \).

This \( c \) is a pseudo-distance on \( D \); \( c \) is a distance if and only if \( B(D) \) separates the points of \( D \), in which case we say that \( D \) is \( c \)-hyperbolic. If every closed ball \( \Delta(p, r) = \{ x \in D; c(p, x) \leq r \}, p \in D \) and \( r > 0 \), is compact, we call \( D \) a \( c \)-complete domain. Horstmann [3] has shown that a \( c \)-complete domain in \( C^n \) is holomorphically convex. Kobayashi [6], [7] has generalized this as follows: a \( c \)-complete domain in a complex space is \( B \)-holomorphically convex. (See [7, Theorem 3.6, Chapter 4].)

We note the following facts about the Carathéodory distance \( c \). \( c \) is trivial on \( C^n \) or on a compact complex space. Every holomorphic map of a complex space to another is distance decreasing. A finite Cartesian product of \( c \)-complete hyperbolic complex spaces is \( c \)-complete hyperbolic. An intersection of \( c \)-complete hyperbolic complex subspaces of a complex space is \( c \)-complete hyperbolic. (See Kobayashi [6], [7].)

We will use the following relatively unknown terminology throughout this note. A domain \( D \) in a complex space \( (X, A) \) is said to be a domain of bounded holomorphy if there is a function \( f \in B(D) \) which does not have bounded analytic continuation beyond the domain \( D \). \( D \) is said to be boundedly holomorphic convex if the holomorphically convex hull \( \tilde{K}_B \) relative to \( B(D) \) \( (\tilde{K}_B = \{ x \in D; |f(x)| \leq \|f\|_K \text{ for all } f \in B(D) \}) \) is compact for every compact subset \( K \) of \( D \). An envelope of bounded holomorphy is the largest domain into which all bounded holomorphic functions may be continued boundedly (see Kim [4, Definition 2 and Theorem 2]). Finally, a Stein manifold of bounded type is a complex manifold \( (X, A) \) such that (i) \( B(X) \) separates the points of \( X \), (ii) \( X \) is boundedly holomorphic convex, and (iii) \( B(X) \) provides a globally defined local coordinate system to each point of \( X \).

**Proposition 1.** Let \( (X_1, A_1) \) and \( (X_2, A_2) \) be \( c \)-hyperbolic complex spaces and \( \phi \) a proper holomorphic map of \( X_1 \) onto \( X_2 \). If \( X_2 \) is \( c \)-complete then so is \( X_1 \).
Proof. Let $c_{X_1}$ and $c_{X_2}$ be the distances on $X_1$ and $X_2$, respectively. Since $c_{X_2}(\phi(p), \phi(x)) \leq c_{X_1}(p, x)$ for $p, x \in X_1$,
$$\{x \in X_1; c_{X_1}(p, x) \leq r\} \subseteq \phi^{-1}(\{y \in X_2; c_{X_2}(\phi(p), y) \leq r\}).$$
Since the latter set is compact, so is the former.

**Theorem 2.** Every bounded convex (in the geometric sense) domain in $\mathbb{C}^n$ ($\mathbb{R}^{2n}$) is $c$-complete.

Proof. Such a domain $D$ is the intersection of open sets biholomorphic to $S = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; \text{Re } z_i > 0, i = 1, 2, \ldots, n\}$. Since such $S$'s are $c$-complete, so is $D$.

**Remark.** We have a large class of domains $D$ on which $B(D)$ is dense in $\widehat{O}(D)$. By the above theorem, every bounded convex domain $D$ in $\mathbb{C}^n$ is boundedly holomorphic convex so that it is a Stein manifold of bounded type, hence $B(D)$ is dense in $\widehat{O}(D)$.

**Proposition 3.** A Siegel domain of the second kind is $c$-complete hyperbolic.

Proof. A Siegel domain of the second kind can be written as the intersection of domains, each of which is biholomorphic to a product of balls. Since a product of balls is $c$-complete hyperbolic, so is the domain.

We note that, in a Riemann domain $(X, A; \alpha)$ with a bounded spread map $\alpha$, if a domain $D$ in $X$ is boundedly holomorphic convex, then $B(D)$ separates the points of $D$ (see Kim [5]), so that such a domain is always $c$-hyperbolic.

To preserve $c$-completeness from one complex space to another, it suffices to have a local biholomorphic proper map.

**Theorem 4.** Let $(X, A)$ and $(\tilde{X}, \tilde{A})$ be $c$-hyperbolic complex spaces. Let $\alpha$ be a proper spread map of $X$ onto $\tilde{X}$. Then $X$ is $c$-complete if and only if $\tilde{X}$ is $c$-complete.

Proof. If $\tilde{X}$ is $c$-complete so is $X$ by Proposition 1. Assume that $X$ is $c$-complete. Let $\Delta(\tilde{p}, r) = \{\tilde{x} \in \tilde{X}; c_{\tilde{X}}(\tilde{p}, \tilde{x}) \leq r\}, \tilde{p} \in \tilde{X}$. We show that $\Delta(\tilde{p}, r)$ is compact. Note that since $\alpha$ is a proper spread map, $\alpha^{-1}(\tilde{x})$ is, for any $\tilde{x} \in \tilde{X}$, a finite point set. For $x \in X$, there is a neighborhood $U_x$ such that $\alpha: U_x \rightarrow \alpha(U_x)$ is biholomorphic. Set $\alpha(U_x) = U_x$. Then there exists $\varepsilon_x > 0$ such that $\Delta(x, \varepsilon_x) = \{y \in \tilde{X}; c_X(x, y) < \varepsilon_x\} \subseteq U_x$. Consider the family $\{\Delta(\tilde{p}, r) \subseteq \tilde{X} \mid x \rightarrow (\tilde{x}, \tilde{y})\}$. This family is an open covering of $\Delta(\tilde{p}, r)$.
Now consider \( \{ \alpha^{-1}(\Delta(\mathcal{X}, e_{\mathcal{X}})) \}; \mathcal{X} \in \Delta(\mathcal{P}, r) \}. \) Recalling that \( \alpha \) is an isometry of each \( \alpha^{-1}(\Delta(\mathcal{X}, e_{\mathcal{X}})) \) to \( \Delta(\mathcal{X}, e_{\mathcal{X}}) \), and that the preimage of \( \mathcal{X} \) is a finite set, we have that \( \bigcup \alpha^{-1}(\Delta(\mathcal{X}, e_{\mathcal{X}})) \) is contained in \( \Delta(p, a) \), for \( \alpha(p) = \mathcal{P} \) and some \( a < \infty \). Since \( \Delta(p, a) \) is compact, choosing a finite covering \( \{ \alpha^{-1}(\Delta(\mathcal{X}, e_{\mathcal{X}})) \}; i = 1, 2, \ldots, n \}, \Delta(\mathcal{P}, r) \) has a finite covering. Hence \( \Delta(\mathcal{P}, r) \) is compact.

The following discussion is limited to Riemann domains.

**Proposition 5.** Let \((X_1, A_1; a_1)\) and \((X_2, A_2; a_2)\) be Riemann domains, and \((\beta_1; \mathcal{X}_1; \mathcal{X}_1, \mathcal{B}_1), (\beta_2; \mathcal{X}_2; \mathcal{X}_2, \mathcal{B}_2)\) the envelopes of bounded holomorphy of \(X_1\) and \(X_2\), respectively. Let \(\phi: X_1 \rightarrow X_2\) be a spread map of \(X_1\) onto \(X_2\). Then there exists a holomorphic map \(\phi: \mathcal{X}_1 \rightarrow \mathcal{X}_2\) such that \(\phi \circ \beta_1 = \beta_2 \circ \phi\).

**Proof.**

Let \(\psi = \beta_2 \circ \phi: X_1 \rightarrow \mathcal{X}_2\). Then \(\psi\) is holomorphic and a local biholomorphism. We will show that there is \(\tilde{\phi}: \mathcal{X}_1 \rightarrow \mathcal{X}_2\) such that \(\tilde{\phi} \circ \beta_1 = \psi\).

Let \(\rho = \tilde{\phi} \circ \psi\); then \(\rho\) is also a local biholomorphism. Let \(J\) be the Jacobian determinant \(J = \det(\partial_{a_i} \rho_j / d\zeta_j)\). Then since \(\rho\) is a local biholomorphism, \(J(x) \neq 0\) for all \(x \in X_1\). Let \(\tilde{\rho}_j\) be the extension of \(\rho_j\) to \(\mathcal{X}_1\), and let \(\tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_n)\). Let \(\tilde{J}\) be the extension of \(J\) to \(\mathcal{X}_1\). Then \(\tilde{J} = \det(\partial_{\tilde{a}_1} \tilde{\rho}_j / d\zeta_j)\) and \(\tilde{J}(\mathcal{X}) \neq 0\) for all \(\mathcal{X} \in \mathcal{X}_1\). Hence \(\tilde{\rho}: \mathcal{X}_1 \rightarrow \mathcal{C}^n\) is a local biholomorphism and \(\tilde{\rho} \circ \beta_1 = \rho\).

Let \(F = \{ f \circ \phi; f \in B(X_2) \}\), and identify this with \(\{ f \circ \psi; \tilde{f} \in B(\mathcal{X}_2) = \mathcal{B}_2 \}\). It follows that \(\{ \tilde{X}_2; \mathcal{X}_2, \mathcal{B}_2 \}\) is the \(F\)-envelope of holomorphy of \(\rho: X_1 \rightarrow \mathcal{C}^n\). Now, any bounded holomorphic function on \(X_1\) can be extended to \(\tilde{\mathcal{X}}_1\) so that \(\tilde{\rho}: \tilde{\mathcal{X}}_1 \rightarrow \mathcal{C}^n\) is an \(F\)-extension of \(\rho: X_1 \rightarrow \mathcal{C}^n\) relative to \(\beta_1: X_1 \rightarrow \tilde{\mathcal{X}}_1\). Since \(\mathcal{X}_2: \tilde{X}_2 \rightarrow \mathcal{C}^n\) is the \(F\)-envelope of holomorphy of \(\rho: X_1 \rightarrow \mathcal{C}^n\), there exists a holomorphic map \(\tilde{\phi}: \tilde{\mathcal{X}}_1 \rightarrow \tilde{\mathcal{X}}_2\) such that \(\mathcal{X}_2 \circ \tilde{\phi} = \tilde{\rho}\) and \(\tilde{\phi} \circ \beta_1 = \psi\).

**Corollary 6.** Let \((X, A; a)\) be a Riemann domain and \((\beta; \mathcal{X}, \mathcal{X}; \tilde{\mathcal{X}}, \mathcal{B})\) its envelope of bounded holomorphy. Then for any analytic automorphism
Proposition 7 (H. Cartan). Let \((X, A; \alpha)\) be a Riemann domain and \(D\) a domain in \(X\). Let \(\{f_\nu\} \subset \text{Aut}(D)\) be a sequence of automorphisms of \(D\). Suppose that \(\{f_\nu\}\) converges uniformly on compact subsets of \(D\) to a holomorphic map \(f: D \to X\). Then the following conditions are equivalent.

(i) \(f \in \text{Aut}(D)\);
(ii) \(f(D) \notin \text{boundary of } D\);
(iii) there exists \(a \in D\) such that the Jacobian of \(f\) at \(a\) is nontrivial.

Theorem 8. Let \((X, A; \alpha)\) be a separable Riemann domain. Let \(D\) be a domain in \(X\) with \(\alpha(D)\) bounded in \(\mathbb{C}^n\). Let \(B(D)\) separate the points of \(D\). Suppose that there is a compact set \(K\) such that for any \(x \in D\) there is an analytic automorphism \(\sigma \in \text{Aut}(D)\) and a point \(a \in K\) with \(\sigma(x) = a\). Then \(D\) is a domain of bounded holomorphy.

Proof. Let \((\beta; \tilde{D}, \tilde{A}; \tilde{\alpha}, \tilde{B})\) be the envelope of bounded holomorphy of \(D\) so that \(\alpha = \tilde{\alpha} \circ \beta\). Then \(\beta\) is injective. To show the assertion, we have to show that \(\beta\) is surjective. Suppose this were false. Let \(\{x_\nu\}\) be a sequence of points of \(D\) which does not have a limit point in \(D\) and such that \(\{\beta(x_\nu)\}\) converges to a point \(q\) in the intersection of the boundary of \(\beta(D)\) and \(\tilde{D}\). Let \(a_\nu \in K\) and \(\sigma_\nu \in \text{Aut}(D)\) be such that \(\sigma_\nu(x_\nu) = a_\nu\). Let \(P\) be an \(\tilde{\alpha}\)-polydisc about \(q \in \tilde{D}\), with \(P\) relatively compact in \(\tilde{D}\) so that \(\tilde{\alpha}\) is biholomorphic on \(P\). By Corollary 6, there is an automorphism \(\tilde{\alpha}_\nu\) of \(\tilde{D}\) such that \(\tilde{\alpha}_\nu \circ \beta = \beta \circ \sigma_\nu\). Further, since \(\alpha(D)\) is bounded, \(\alpha\) is a bounded spread map on \(D\), that is, \(\alpha = (f_1, \ldots, f_n)\) with \(f_j\) bounded, so that \(\tilde{\alpha} = (\tilde{f}_1, \ldots, \tilde{f}_n)\) is bounded on \(\tilde{D}\) and \(\tilde{\alpha} \circ \tilde{\alpha}_\nu\) is bounded uniformly with respect to \(\nu\). Let \(P_\rho\) be the polydisc of radius \(\rho\) about \(q\) in \(P\). Then there is a constant \(c_\rho > 0\) such that for \(y \in P_\rho\), \(|\tilde{\alpha}_\nu(x) - \tilde{\alpha}_\nu(y)| \leq c_\rho\) for all \(x \in P_\rho\). Since \(\beta\) is injective, it follows that for sufficiently small \(\rho\) there is a compact subset \(L\) of \(D\) such that \(\sigma_\nu(\beta^{-1}(P_\rho \cap \beta(D))) = \sigma_\nu(\beta^{-1}(P_\rho)) \subset L\).

By passing to subsequences, let \(\sigma\) and \(\sigma': D \to \mathbb{C}^n\) be the uniform limits of \(\{\sigma_\nu\}\) and \(\{\sigma^{-1}_\nu\}\) on compact subsets of \(D\). Hence by Proposition 7, \(\sigma\), \(\sigma' \in \text{Aut}(D)\) and \(\sigma' \circ \sigma = \sigma \circ \sigma' = \text{identity}\). However, this is absurd, since \(\sigma^{-1}(a_\nu) = x_\nu\), so that if \(a\) is a limit point of \(\{a_\nu\}\) in \(K\), \(\sigma'(a) \in D\), but \(\{x_\nu\}\) has no limit point in \(D\). The theorem is proved.

Corollary 9. If \(\Gamma\) is a discrete subgroup of \(\text{Aut}(D)\) such that \(D/\Gamma\) is
compact, then \( D \) is a domain of bounded holomorphy.

**Corollary 10.** If \( D \) is a bounded homogeneous domain in \( \mathbb{C}^n \), then \( D \) is a domain of bounded holomorphy.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611