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COMPLETE DOMAINS WITH RESPECT TO THE CARATHÉODORY DISTANCE

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ABSTRACT. Concerning completeness with respect to the Carathéodory distance (c-completeness), the following theorems are shown. A bounded convex (in geometric sense) domain D in C^n (R^{2n}) is c-complete, so that it is boundedly holomorphic convex. To preserve c-completeness in complex spaces, it is sufficient to have a proper local biholomorphic mapping as follows: Let α be a proper spread map of a c-hyperbolic complex space (X, A) onto a c-hyperbolic complex space $(\tilde{X}, \tilde{A})$; then X is c-complete if and only if $\tilde{X}$ is c-complete. We also show the following D to be domains of bounded holomorphy: let $(X, A; \alpha)$ be a Riemann domain and D a domain in X with $\alpha(D)$ bounded in C^n. Let $B(D)$ separate the points of D. Suppose there is a compact set K such that for any $x \in D$ there is an analytic automorphism $\sigma \in \mathrm{Aut}(D)$ and a point $a \in K$ such that $\sigma(x) = a$. Then D is a domain of bounded holomorphy.

Let (X, A) be a complex space and D a domain (open and connected) in X. Let $B = B(D)$ be the algebra of bounded holomorphic functions on D and

$$B_1 = \left\{ f \in B;\ \sup_{x \in D} |f(x)| = \|f\|_D = 1 \right\}.$$

We define the Carathéodory distance $c = c_D$ as follows: For $x, y \in D$,

$$c(x, y) = \sup_{g \in B_1} \rho(g(x), g(y)),$$

where

$$\rho(z_1, z_2) = \log \frac{|z_2 - z_1| + |1 - z_1\bar{z}_2|}{\sqrt{(1 - z_1\bar{z}_1)(1 - z_2\bar{z}_2)}},$$

where z_1, z_2 are in the open unit disc in $\mathbf{C}$.

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For $g \in B_1$ and $x' \in D$, set

$$f(x') = \frac{g(x') - g(x)}{g(x')\overline{g(x)} - 1};$$

then

$$c(x, y) = \sup_{f \in B_x} \left\{ \frac{1}{2} \log \frac{1 + |f(y)|}{1 - |f(y)|} \right\},$$

where $B_x = \{f \in B_1;\ f(x) = 0\}$.

This c is a pseudo-distance on D; c is a distance if and only if $B(D)$ separates the points of D, in which case we say that D is c-hyperbolic. If every closed ball $\Delta(p, r) = \{x \in D;\ c(p, x) \leq r\}$, $p \in D$ and $r > 0$, is compact, we call D a c-complete domain. Horstmann [3] has shown that a c-complete domain in $\mathbf{C}^n$ is holomorphically convex. Kobayashi [6], [7] has generalized this as follows: a c-complete domain in a complex space is B-holomorphically convex. (See [7, Theorem 3.6, Chapter 4].)

We note the following facts about the Carathéodory distance c. c is trivial on $\mathbf{C}^n$ or on a compact complex space. Every holomorphic map of a complex space to another is distance decreasing. A finite Cartesian product of c-complete hyperbolic complex spaces is c-complete hyperbolic. An intersection of c-complete hyperbolic complex subspaces of a complex space is c-complete hyperbolic. (See Kobayashi [6], [7].)

We will use the following relatively unknown terminology throughout this note. A domain D in a complex space (X, A) is said to be a domain of bounded holomorphy if there is a function $f \in B(D)$ which does not have bounded analytic continuation beyond the domain D. D is said to be boundedly holomorphic convex if the holomorphically convex hull $\hat{K}_B$ relative to $B(D)$ ($\hat{K}_B = \{x \in D;\ |f(x)| \leq \|f\|_K$ for all $f \in B(D)\}$) is compact for every compact subset K of D. An envelope of bounded holomorphy is the largest domain into which all bounded holomorphic functions may be continued boundedly (see Kim [4, Definition 2 and Theorem 2]). Finally, a Stein manifold of bounded type is a complex manifold (X, A) such that (i) $B(X)$ separates the points of X, (ii) X is boundedly holomorphic convex, and (iii) $B(X)$ provides a globally defined local coordinate system to each point of X.

Proposition 1. *Let (X_1, A_1) and (X_2, A_2) be c-hyperbolic complex spaces and ϕ a proper holomorphic map of X_1 onto X_2. If X_2 is c-complete then so is X_1.*

Proof. Let c_{X_1} and c_{X_2} be the distances on X_1 and X_2, respectively. Since $c_{X_2}(\phi(p), \phi(x)) \leq c_{X_1}(p, x)$ for $p, x \in X_1$,

$$\{x \in X_1;\ c_{X_1}(p, x) \leq r\} \subset \phi^{-1}(\{y \in X_2;\ c_{X_2}(\phi(p), y) \leq r\}).$$

Since the latter set is compact, so is the former.

Theorem 2. *Every bounded convex (in the geometric sense) domain in* $\mathbf{C}^n$ ($\mathbf{R}^{2n}$) *is c-complete.*

Proof. Such a domain D is the intersection of open sets biholomorphic to $S = \{(z_1, \cdots, z_n) \in \mathbf{C}^n;\ \mathrm{Re}\, z_i > 0,\ i = 1, 2, \cdots, n\}$. Since such S's are c-complete, so is D.

Remark. We have a large class of domains D on which $B(D)$ is dense in $\mathcal{O}(D)$. By the above theorem, every bounded convex domain D in $\mathbf{C}^n$ is boundedly holomorphic convex so that it is a Stein manifold of bounded type, hence $B(D)$ is dense in $\mathcal{O}(D)$.

Proposition 3. *A Siegel domain of the second kind is c-complete hyperbolic.*

Proof. A Siegel domain of the second kind can be written as the intersection of domains, each of which is biholomorphic to a product of balls. Since a product of balls is c-complete hyperbolic, so is the domain.

We note that, in a Riemann domain $(X, A; \alpha)$ with a bounded spread map α, if a domain D in X is boundedly holomorphic convex, then $B(D)$ separates the points of D (see Kim [5]), so that such a domain is always c-hyperbolic.

To preserve c-completeness from one complex space to another, it suffices to have a local biholomorphic proper map.

Theorem 4. *Let* (X, A) *and* $(\tilde{X}, \tilde{A})$ *be c-hyperbolic complex spaces. Let* α *be a proper spread map of* X *onto* $\tilde{X}$. *Then* X *is c-complete if and only if* $\tilde{X}$ *is c-complete.*

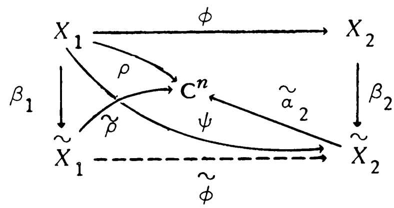
Proof. If $\tilde{X}$ is c-complete so is X by Proposition 1. Assume that X is c-complete. Let $\Delta(\tilde{p}, r) = \{\tilde{x} \in \tilde{X};\ c_{\tilde{X}}(\tilde{p}, \tilde{x}) \leq r\}$, $\tilde{p} \in \tilde{X}$. We show that $\Delta(\tilde{p}, r)$ is compact. Note that since α is a proper spread map, $\alpha^{-1}(\tilde{x})$ is, for any $\tilde{x} \in \tilde{X}$, a finite point set. For $x \in X$, there is a neighborhood U_x such that $\alpha\colon U_x \to \alpha(U_x)$ is biholomorphic. Set $\alpha(U_x) = U_{\tilde{X}}$. Then there exists $\epsilon_{\tilde{x}} > 0$ such that $\mathring{\Delta}(\tilde{x}, \epsilon_{\tilde{x}}) = \{\tilde{y} \in \tilde{X};\ c_{\tilde{X}}(\tilde{x}, \tilde{y}) < \epsilon_{\tilde{x}}\} \subset U_{\tilde{x}}$. Consider the family $\{\mathring{\Delta}(\tilde{x}, \epsilon_{\tilde{x}});\ \tilde{x} \in \Delta(\tilde{p}, r)\}$. This family is an open covering of $\Delta(\tilde{p}, r)$.

Now consider $\{\alpha^{-1}(\mathring{\Delta}(\tilde{x}, \epsilon_{\tilde{x}})); \tilde{x} \in \Delta(\tilde{p}, r)\}$. Recalling that α is an isometry of each $\alpha^{-1}(\mathring{\Delta}(\tilde{x}, \epsilon_{\tilde{x}}))$ to $\mathring{\Delta}(\tilde{x}, \epsilon_{\tilde{x}})$, and that the preimage of $\tilde{x}$ is a finite set, we have that $\bigcup \alpha^{-1}(\mathring{\Delta}(\tilde{x}, \epsilon_{\tilde{x}}))$ is contained in $\Delta(p, a)$, for $\alpha(p) = \tilde{p}$ and some $a < \infty$. Since $\Delta(p, a)$ is compact, choosing a finite covering $\{\alpha^{-1}(\mathring{\Delta}(\tilde{x}_i, \epsilon_{\tilde{x}_i})); i = 1, 2, \cdots, n\}$, $\Delta(\tilde{p}, r)$ has a finite covering. Hence $\Delta(\tilde{p}, r)$ is compact.

The following discussion is limited to Riemann domains.

Proposition 5. *Let $(X_1, A_1; \alpha_1)$ and $(X_2, A_2; \alpha_2)$ be Riemann domains, and $(\beta_1; \tilde{X}_1, \tilde{A}_1; \tilde{\alpha}_1, \tilde{B}_1)$, $(\beta_2; \tilde{X}_2, \tilde{A}_2; \tilde{\alpha}_2, \tilde{B}_2)$ the envelopes of bounded holomorphy of X_1 and X_2, respectively. Let $\phi: X_1 \to X_2$ be a spread map of X_1 onto X_2. Then there exists a holomorphic map $\tilde{\phi}: \tilde{X}_1 \to \tilde{X}_2$ such that $\tilde{\phi} \circ \beta_1 = \beta_2 \circ \phi$.*

Proof.



Let $\psi = \beta_2 \circ \phi: X_1 \to \tilde{X}_2$. Then ψ is holomorphic and a local biholomorphism. We will show that there is $\tilde{\phi}: \tilde{X}_1 \to \tilde{X}_2$ such that $\tilde{\phi} \circ \beta_1 = \psi$. Let $\rho = \tilde{\alpha}_2 \circ \psi$; then ρ is also a local biholomorphism. Let $\rho = (\rho_1, \cdots, \rho_n)$, ρ_i holomorphic. Let J be the jacobian determinant $J = \det(\partial_{a_1}\rho_i/\partial z_j)$. Then since ρ is a local biholomorphism, $J(x) \neq 0$ for all $x \in X_1$. Let $\tilde{\rho}_j$ be the extension of ρ_j to $\tilde{X}_1$, and let $\tilde{\rho} = (\tilde{\rho}_1, \cdots, \tilde{\rho}_n)$. Let $\tilde{J}$ be the extension of J to $\tilde{X}_1$. Then $\tilde{J} = \det(\partial_{\tilde{\alpha}_1}\tilde{\rho}_i/\partial z_j)$ and $\tilde{J}(\tilde{x}) \neq 0$ for all $\tilde{x} \in \tilde{X}_1$. Hence $\tilde{\rho}: \tilde{X}_1 \to \mathbf{C}^n$ is a local biholomorphism and $\tilde{\rho} \circ \beta_1 = \rho$.

Let $F = \{f \circ \phi; f \in B(X_2)\}$, and identify this with $\{\tilde{f} \circ \psi; \tilde{f} \in B(\tilde{X}_2) = \tilde{B}_2\}$. It follows that $\{\tilde{X}_2; \tilde{\alpha}_2, \tilde{B}_2\}$ is the F-envelope of holomorphy of ρ: $X_1 \to \mathbf{C}^n$. Now, any bounded holomorphic function on X_1 can be extended to $\tilde{X}_1$ so that $\tilde{\rho}: \tilde{X}_1 \to \mathbf{C}^n$ is an F-extension of $\rho: X_1 \to \mathbf{C}^n$ relative to $\beta_1: X_1 \to \tilde{X}_1$. Since $\tilde{\alpha}_2: \tilde{X}_2 \to \mathbf{C}^n$ is the F-envelope of holomorphy of ρ: $X_1 \to \mathbf{C}^n$, there exists a holomorphic map $\tilde{\phi}: \tilde{X}_1 \to \tilde{X}_2$ such that $\tilde{\alpha}_2 \circ \tilde{\phi} = \tilde{\rho}$ and $\tilde{\phi} \circ \beta_1 = \psi$.

Corollary 6. *Let $(X, A; \alpha)$ be a Riemann domain and $(\beta; \tilde{X}, \tilde{A}; \tilde{\alpha}, \tilde{B})$ its envelope of bounded holomorphy. Then for any analytic automorphism*

σ of X, there exists an analytic automorphism $\tilde{\sigma}$ of $\tilde{X}$ such that $\tilde{\sigma} \circ \beta = \beta \circ \sigma$.

Proposition 7 (H. Cartan). *Let $(X, A; \alpha)$ be a Riemann domain and D a domain in X. Let $\{f_\nu\} \subset \operatorname{Aut}(D)$ be a sequence of automorphisms of D. Suppose that $\{f_\nu\}$ converges uniformly on compact subsets of D to a holomorphic map f: $D \to X$. Then the following conditions are equivalent.*

(i) $f \in \operatorname{Aut}(D)$;

(ii) $f(D) \not\subset$ *boundary of* D;

(iii) *there exists $a \in D$ such that the jacobian of f at a is nontrivial.*

Theorem 8. *Let $(X, A; \alpha)$ be a separable Riemann domain. Let D be a domain in X with $\alpha(D)$ bounded in $\mathbf{C}^n$. Let $B(D)$ separate the points of D. Suppose that there is a compact set K such that for any $x \in D$ there is an analytic automorphism $\sigma \in \operatorname{Aut}(D)$ and a point $a \in K$ with $\sigma(x) = a$. Then D is a domain of bounded holomorphy.*

Proof. Let $(\beta; \tilde{D}, \tilde{X}; \tilde{\alpha}, \tilde{B})$ be the envelope of bounded holomorphy of D so that $\alpha = \tilde{\alpha} \circ \beta$. Then β is injective. To show the assertion, we have to show that β is surjective. Suppose this were false. Let $\{x_\nu\}$ be a sequence of points of D which does not have a limit point in D and such that $\{\beta(x_\nu)\}$ converges to a point q in the intersection of the boundary of $\beta(D)$ and $\tilde{D}$. Let $a_\nu \in K$ and $\sigma_\nu \in \operatorname{Aut}(D)$ be such that $\sigma_\nu(x_\nu) = a_\nu$. Let $\mathbf{P}$ be an $\tilde{\alpha}$-polydisc about $q \in \tilde{D}$, with $\mathbf{P}$ relatively compact in $\tilde{D}$ so that $\tilde{\alpha}$ is biholomorphic on $\mathbf{P}$. By Corollary 6, there is an automorphism $\tilde{\sigma}_\nu$ of $\tilde{D}$ such that $\tilde{\sigma}_\nu \circ \beta = \beta \circ \sigma_\nu$. Further, since $\alpha(D)$ is bounded, α is a bounded spread map on D, that is, $\alpha = (f_1, \cdots, f_n)$ with f_j bounded, so that $\tilde{\alpha} = (\tilde{f}_1, \cdots, \tilde{f}_n)$ is bounded on $\tilde{D}$ and $\tilde{\alpha} \circ \tilde{\sigma}_\nu$ is bounded uniformly with respect to ν. Let $\mathbf{P}_\rho$ be the polydisc of radius ρ about q in $\mathbf{P}$. Then there is a constant $c_\rho > 0$ such that for $y \in \mathbf{P}_\rho$, $|\tilde{\sigma}_\nu(x) - \tilde{\sigma}_\nu(y)| \leq c_\rho$ for all $x \in \mathbf{P}_\rho$. Since β is injective, it follows that for sufficiently small ρ there is a compact subset L of D such that $\sigma_\nu(\beta^{-1}(\mathbf{P}_\rho \cap \beta(D))) = \sigma_\nu(\beta^{-1}(\mathbf{P}_\rho)) \subset L$.

By passing to subsequences, let σ and σ': $D \to \mathbf{C}^n$ be the uniform limits of $\{\sigma_\nu\}$ and $\{\sigma_\nu^{-1}\}$ on compact subsets of D. Hence by Proposition 7, σ, $\sigma' \in \operatorname{Aut}(D)$ and $\sigma' \circ \sigma = \sigma \circ \sigma' =$ identity. However, this is absurd, since $\sigma_\nu^{-1}(a_\nu) = x_\nu$, so that if a is a limit point of $\{a_\nu\}$ in K, $\sigma'(a) \in D$, but $\{x_\nu\}$ has no limit point in D. The theorem is proved.

Corollary 9. *If Γ is a discrete subgroup of* Aut(D) *such that D/Γ is* *compact, then D is a domain of bounded holomorphy.*

Corollary 10. *If D is a bounded homogeneous domain in* $\mathbf{C}^n$, *then D is a domain of bounded holomorphy.*

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