COMPLETE DOMAINS WITH RESPECT TO THE CARATHÉODORY DISTANCE

DONG S. KIM

ABSTRACT. Concerning completeness with respect to the Carathéodory distance (c-completeness), the following theorems are shown. A bounded convex (in geometric sense) domain $D$ in $\mathbb{C}^n$ ($\mathbb{R}^{2n}$) is $c$-complete, so that it is boundedly holomorphic convex. To preserve $c$-completeness in complex spaces, it is sufficient to have a proper local biholomorphic mapping as follows: Let $\alpha$ be a proper spread map of a $c$-hyperbolic complex space $(X, A)$ onto a $c$-hyperbolic complex space $(\tilde{X}, \tilde{A})$; then $X$ is $c$-complete if and only if $\tilde{X}$ is $c$-complete. We also show the following $D$ to be domains of bounded holomorphy: let $(X, A)$ be a Riemann domain and $D$ a domain in $X$ with $\alpha(D)$ bounded in $\mathbb{C}^n$. Let $B(D)$ separate the points of $D$. Suppose there is a compact set $K$ such that for any $x \in D$ there is an analytic automorphism $\sigma \in \text{Aut}(D)$ and a point $a \in K$ such that $\sigma(x) = a$. Then $D$ is a domain of bounded holomorphy.

Let $(X, A)$ be a complex space and $D$ a domain (open and connected) in $X$. Let $B = B(D)$ be the algebra of bounded holomorphic functions on $D$ and

$$B_1 = \left\{ f \in B; \sup_{x \in D} |f(x)| = \|f\|_D = 1 \right\}.$$ 

We define the Carathéodory distance $c = c_D$ as follows: For $x, y \in D$,

$$c(x, y) = \sup_{g \in B_1} \rho(g(x), g(y)),$$

where

$$\rho(z_1, z_2) = \log \frac{|z_2 - z_1| + |1 - z_1\overline{z_2}|}{\sqrt{(1 - z_1\overline{z_1})(1 - z_2\overline{z_2})}},$$

where $z_1, z_2$ are in the open unit disc in $\mathbb{C}$.

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For $g \in B_1$ and $x' \in D$, set

$$f(x') = \frac{g(x') - g(x)}{g(x')g(x) - 1};$$

then

$$c(x, y) = \sup_{f \in B_x} \left\{ \frac{1}{2} \log \frac{1 + \|f(y)\|}{1 - \|f(y)\|} \right\},$$

where $B_x = \{f \in B_1; f(x) = 0\}$.

This $c$ is a pseudo-distance on $D$; $c$ is a distance if and only if $B(D)$ separates the points of $D$, in which case we say that $D$ is $c$-hyperbolic. If every closed ball $\Delta(p, r) = \{x \in D; c(p, x) \leq r\}, p \in D$ and $r > 0$, is compact, we call $D$ a $c$-complete domain. Horstmann [3] has shown that a $c$-complete domain in $\mathbb{C}^n$ is holomorphically convex. Kobayashi [6], [7] has generalized this as follows: a $c$-complete domain in a complex space is $B$-holomorphically convex. (See [7, Theorem 3.6, Chapter 4].)

We note the following facts about the Carathéodory distance $c$. $c$ is trivial on $\mathbb{C}^n$ or on a compact complex space. Every holomorphic map of a complex space to another is distance decreasing. A finite Cartesian product of $c$-complete hyperbolic complex spaces is $c$-complete hyperbolic. An intersection of $c$-complete hyperbolic complex subspaces of a complex space is $c$-complete hyperbolic. (See Kobayashi [6], [7].)

We will use the following relatively unknown terminology throughout this note. A domain $D$ in a complex space $(X, A)$ is said to be a domain of bounded holomorphy if there is a function $f \in B(D)$ which does not have bounded analytic continuation beyond the domain $D$. $D$ is said to be boundedly holomorphic convex if the holomorphically convex hull $\hat{K}_B$ relative to $B(D)$ ($\hat{K}_B = \{x \in D; \|f(x)\| \leq \|f\|_K$ for all $f \in B(D)\}$) is compact for every compact subset $K$ of $D$. An envelope of bounded holomorphy is the largest domain into which all bounded holomorphic functions may be continued boundedly (see Kim [4, Definition 2 and Theorem 2]). Finally, a Stein manifold of bounded type is a complex manifold $(X, A)$ such that (i) $B(X)$ separates the points of $X$, (ii) $X$ is boundedly holomorphic convex, and (iii) $B(X)$ provides a globally defined local coordinate system to each point of $X$.

**Proposition 1.** Let $(X_1, A_1)$ and $(X_2, A_2)$ be $c$-hyperbolic complex spaces and $\phi$ a proper holomorphic map of $X_1$ onto $X_2$. If $X_2$ is $c$-complete then so is $X_1$. 
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Proof. Let $c_{X_1}$ and $c_{X_2}$ be the distances on $X_1$ and $X_2$, respectively. Since $c_{X_2} (\phi(p), \phi(x)) \leq c_{X_1} (p, x)$ for $p, x \in X_1$,

$$\{x \in X_1; c_{X_1} (p, x) \leq r\} \subset \phi^{-1} (\{y \in X_2; c_{X_2} (\phi(p), y) \leq r\}).$$

Since the latter set is compact, so is the former.

Theorem 2. Every bounded convex (in the geometric sense) domain in $\mathbb{C}^n (\mathbb{R}^{2n})$ is c-complete.

Proof. Such a domain $D$ is the intersection of open sets biholomorphic to $S = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; \text{Re } z_i > 0, i = 1, 2, \ldots, n\}$. Since such $S$'s are c-complete, so is $D$.

Remark. We have a large class of domains $D$ on which $B(D)$ is dense in $\mathcal{O}(D)$. By the above theorem, every bounded convex domain $D$ in $\mathbb{C}^n$ is boundedly holomorphic convex so that it is a Stein manifold of bounded type, hence $B(D)$ is dense in $\mathcal{O}(D)$.

Proposition 3. A Siegel domain of the second kind is c-complete hyperbolic.

Proof. A Siegel domain of the second kind can be written as the intersection of domains, each of which is biholomorphic to a product of balls. Since a product of balls is c-complete hyperbolic, so is the domain.

We note that, in a Riemann domain $(X, A; \alpha)$ with a bounded spread map $\alpha$, if a domain $D$ in $X$ is boundedly holomorphic convex, then $B(D)$ separates the points of $D$ (see Kim [5]), so that such a domain is always c-hyperbolic.

To preserve c-completeness from one complex space to another, it suffices to have a local biholomorphic proper map.

Theorem 4. Let $(X, A)$ and $(\tilde{X}, \tilde{A})$ be c-hyperbolic complex spaces. Let $\alpha$ be a proper spread map of $X$ onto $\tilde{X}$. Then $X$ is c-complete if and only if $\tilde{X}$ is c-complete.

Proof. If $\tilde{X}$ is c-complete so is $X$ by Proposition 1. Assume that $X$ is c-complete. Let $\Delta(\tilde{p}, r) = \{\tilde{x} \in \tilde{X}; c_{\tilde{X}} (\tilde{p}, \tilde{x}) \leq r\}, \tilde{p} \in \tilde{X}$. We show that $\Delta(\tilde{p}, r)$ is compact. Note that since $\alpha$ is a proper spread map, $\alpha^{-1}(\tilde{x})$ is, for any $\tilde{x} \in \tilde{X}$, a finite point set. For $x \in X$, there is a neighborhood $U_x$ such that $\alpha: U_x \to \alpha(U_x)$ is biholomorphic. Set $\alpha(U_x) = U_{\tilde{X}}$. Then there exists $\epsilon > 0$ such that $\Delta(\tilde{x}, \epsilon) = \{\tilde{y} \in \tilde{X}; c_{\tilde{X}} (\tilde{x}, \tilde{y}) < \epsilon \} \subset U_{\tilde{X}}$. Consider the family $\{\Delta(\tilde{x}, \epsilon); \tilde{x} \in \Delta(\tilde{p}, r)\}$. This family is an open covering of $\Delta(\tilde{p}, r)$. 


Now consider $\{a^{-1}(\Delta(x', e_{x'})) \mid x' \in \Delta(\beta', r)\}$. Recalling that $a$ is an isometry of each $a^{-1}(\Delta(x', e_{x'}))$ to $\Delta(x', e_{x'})$, and that the preimage of $x'$ is a finite set, we have that $\bigcup a^{-1}(\Delta(x', e_{x'}))$ is contained in $\Delta(\beta, a)$, for $a(\beta) = \beta'$ and some $a < \infty$. Since $\Delta(\beta, a)$ is compact, choosing a finite covering $\{a^{-1}(\Delta(x', e_{x'})) \mid i = 1, 2, \ldots, n\}$, $\Delta(\beta', r)$ has a finite covering. Hence $\Delta(\beta', r)$ is compact.

The following discussion is limited to Riemann domains.

**Proposition 5.** Let $(X_1, A_1, \alpha_1)$ and $(X_2, A_2, \alpha_2)$ be Riemann domains, and $(\beta_1; \tilde{X}_1, \tilde{A}_1, \tilde{B}_1), (\beta_2; \tilde{X}_2, \tilde{A}_2, \tilde{B}_2)$ the envelopes of bounded holomorphy of $X_1$ and $X_2$, respectively. Let $\phi: X_1 \rightarrow X_2$ be a spread map of $X_1$ onto $X_2$. Then there exists a holomorphic map $\tilde{\phi}: \tilde{X}_1 \rightarrow \tilde{X}_2$ such that $\tilde{\phi} \circ \beta_1 = \beta_2 \circ \phi$.

**Proof.**

![Diagram](image)

Let $\psi = \beta_2 \circ \phi: X_1 \rightarrow \tilde{X}_2$. Then $\psi$ is holomorphic and a local biholomorphism. We will show that there is $\tilde{\phi}: \tilde{X}_1 \rightarrow \tilde{X}_2$ such that $\tilde{\phi} \circ \beta_1 = \psi$. Let $\rho = \alpha_2 \circ \psi$; then $\rho$ is also a local biholomorphism. Let $J$ be the jacobian determinant $J = \det(\partial a^i / \partial z_j)$. Then since $\rho$ is a local biholomorphism, $J(x) \neq 0$ for all $x \in X_1$. Let $\tilde{\rho}_j$ be the extension of $\rho_j$ to $\tilde{X}_1$, and let $\tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_n)$. Let $\tilde{J}$ be the extension of $J$ to $\tilde{X}_1$. Then $\tilde{J} = \det(\partial \tilde{\rho}_i / \partial z_j)$ and $\tilde{J}(\tilde{x}) \neq 0$ for all $\tilde{x} \in \tilde{X}_1$. Hence $\tilde{\rho}: \tilde{X}_1 \rightarrow \mathbb{C}^n$ is a local biholomorphism and $\tilde{\rho} \circ \beta_1 = \rho$.

Let $F = \{f \circ \phi; f \in B(X_2)\}$, and identify this with $\{f \circ \psi; \tilde{f} \in B(\tilde{X}_1) = \tilde{B}_2\}$. It follows that $\{\tilde{X}_2, \tilde{\alpha}_2, \tilde{B}_2\}$ is the $F$-envelope of holomorphy of $\rho: X_1 \rightarrow \mathbb{C}^n$. Now, any bounded holomorphic function on $X_1$ can be extended to $\tilde{X}_1$ so that $\tilde{\rho}: \tilde{X}_1 \rightarrow \mathbb{C}^n$ is an $F$-extension of $\rho: X_1 \rightarrow \mathbb{C}^n$ relative to $\beta_1: X_1 \rightarrow \tilde{X}_1$. Since $\tilde{\alpha}_2: \tilde{X}_2 \rightarrow \mathbb{C}^n$ is the $F$-envelope of holomorphy of $\rho: X_1 \rightarrow \mathbb{C}^n$, there exists a holomorphic map $\tilde{\phi}: \tilde{X}_1 \rightarrow \tilde{X}_2$ such that $\tilde{\alpha}_2 \circ \tilde{\phi} = \tilde{\rho}$ and $\tilde{\phi} \circ \beta_1 = \psi$.

**Corollary 6.** Let $(X, A; \alpha)$ be a Riemann domain and $(\beta; \tilde{X}, \tilde{A}; \tilde{\alpha}, \tilde{B})$ its envelope of bounded holomorphy. Then for any analytic automorphism
Proposition 7 (H. Cartan). Let \((X, A; \alpha)\) be a Riemann domain and \(D\) a domain in \(X\). Let \(\{f_{\nu}\} \subset \text{Aut}(D)\) be a sequence of automorphisms of \(D\). Suppose that \(\{f_{\nu}\}\) converges uniformly on compact subsets of \(D\) to a holomorphic map \(f: D \to X\). Then the following conditions are equivalent.

(i) \(f \in \text{Aut}(D)\);
(ii) \(f(D) \notin \text{boundary of } D\);
(iii) there exists \(a \in D\) such that the jacobian of \(f\) at \(a\) is nontrivial.

Theorem 8. Let \((X, A; \alpha)\) be a separable Riemann domain. Let \(D\) be a domain in \(X\) with \(\alpha(D)\) bounded in \(\mathbb{C}^n\). Let \(B(D)\) separate the points of \(D\). Suppose that there is a compact set \(K\) such that for any \(x \in D\) there is an analytic automorphism \(\sigma \in \text{Aut}(D)\) and a point \(a \in K\) with \(\sigma(x) = a\). Then \(D\) is a domain of bounded holomorphy.

Proof. Let \((\beta; D, \tilde{\alpha}; \tilde{\beta}, \tilde{B})\) be the envelope of bounded holomorphy of \(D\) so that \(\alpha = \tilde{\alpha} \circ \beta\). Then \(\beta\) is injective. To show the assertion, we have to show that \(\beta\) is surjective. Suppose this were false. Let \(\{x_{\nu}\}\) be a sequence of points of \(D\) which does not have a limit point in \(D\) and such that \(\{\beta(x_{\nu})\}\) converges to a point \(q\) in the intersection of the boundary of \(\beta(D)\) and \(\tilde{D}\). Let \(a_{\nu} \in K\) and \(\sigma_{\nu} \in \text{Aut}(D)\) be such that \(\sigma_{\nu}(x_{\nu}) = a_{\nu}\). Let \(P\) be an \(\tilde{\alpha}\)-polydisc about \(q \in \tilde{D}\), with \(P\) relatively compact in \(\tilde{D}\) so that \(\tilde{\alpha}\) is biholomorphic on \(P\). By Corollary 6, there is an automorphism \(\tilde{\sigma}_{\nu}\) of \(\tilde{D}\) such that \(\tilde{\sigma}_{\nu} \circ \beta = \beta \circ \sigma_{\nu}\). Further, since \(\alpha(D)\) is bounded, \(\alpha\) is a bounded spread map on \(D\), that is, \(\alpha = (f_1, \ldots, f_n)\) with \(f_j\) bounded, so that \(\tilde{\alpha} = (\tilde{f}_1, \ldots, \tilde{f}_n)\) is bounded on \(\tilde{D}\) and \(\tilde{\alpha} \circ \tilde{\sigma}_{\nu}\) is bounded uniformly with respect to \(\nu\). Let \(P_\rho\) be the polydisc of radius \(\rho\) about \(q\) in \(P\). Then there is a constant \(c_\rho > 0\) such that for \(y \in P_\rho\), \(|\tilde{\sigma}_{\nu}(x) - \tilde{\sigma}_{\nu}(y)| \leq c_\rho\) for all \(x \in P_\rho\). Since \(\beta\) is injective, it follows that for sufficiently small \(\rho\) there is a compact subset \(L\) of \(D\) such that \(\sigma_{\nu}(\beta^{-1}(P_\rho \cap \beta(D))) = \sigma_{\nu}(\beta^{-1}(P_\rho)) \subset L\).

By passing to subsequences, let \(\sigma\) and \(\sigma'\) \(D \to \mathbb{C}^n\) be the uniform limits of \(\{\sigma_{\nu}\}\) and \(\{\sigma_{\nu}^{-1}\}\) on compact subsets of \(D\). Hence by Proposition 7, \(\sigma, \sigma' \in \text{Aut}(D)\) and \(\sigma' \circ \sigma = \sigma \circ \sigma' = \text{identity}\). However, this is absurd, since \(\sigma_{\nu}^{-1}(a_{\nu}) = x_{\nu}\), so that if \(a\) is a limit point of \(\{a_{\nu}\}\) in \(K\), \(\sigma'(a) \in D\), but \(\{x_{\nu}\}\) has no limit point in \(D\). The theorem is proved.

Corollary 9. If \(\Gamma\) is a discrete subgroup of \(\text{Aut}(D)\) such that \(D/\Gamma\) is
compact, then $D$ is a domain of bounded holomorphy.

**Corollary 10.** If $D$ is a bounded homogeneous domain in $\mathbb{C}^n$, then $D$ is a domain of bounded holomorphy.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611