\(\phi\)-POSTNIKOV SYSTEMS AND EXTENSIONS OF \(H\)-SPACES

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ABSTRACT. Let \(f: X \to Y\) be a map of CW complexes and let \(\pi: F \to X\) be the fibration induced by \(f\).

The following theorems are proven:

**Theorem.** Assume \(F (= F_f)\) and \(\Omega Y\) are simply connected and that
(a) \(f^* : H^n(Y; \pi_n(Y)) \to H^n(X; \pi_n(Y))\) is epic for all \(n\),
(b) \((i \wedge i)^* : H^n(F \wedge F; \text{coker } f_n) \to H^n(\Omega Y \wedge \Omega Y; \text{coker } f_n)\) is monic for all \(n\) (where \(f_n : \pi_n(X) \to \pi_n(Y)\)).

If \(X\) is an \(H\)-space then \(F\) is an \(H\)-space such that \(\pi : F \to X\) is an \(H\)-map.

**Theorem.** Assume \(Y\) is \((p - 1)\)-connected, \(F\) is \((q - 1)\)-connected \((p - 1 \geq 2, q - 1 \geq 1)\) of dimension \(< \min(2p - 1, p + q - 1)\), and \(H^*(Y)\) is free. If \(X\) is an \(H\)-space and \(f^* : H^n(Y) \to H^n(X)\) is onto for all \(n\) then \(F\) is an \(H\)-space and the map \(\pi : F \to X\) is an \(H\)-map.

Analogous theorems are shown to hold for loop spaces.

Let \(f: X \to Y\) be a (based) map of CW spaces and let \(\Omega Y \xrightarrow{i} F \xrightarrow{\pi} X\) be the fibration induced by \(f\). That is, \(F_f = \{(x, \eta) \in X \times PY | f(x) = \eta(1)\}\) and \(\pi(x, \eta) = x\).

In this paper we will develop a Postnikov-type decomposition which will allow us to prove

**Theorem 0.1.** Assume \(F (= F_f)\) and \(\Omega Y\) are simply connected and that
(a) \(f^* : H^n(Y; \pi_n(Y)) \to H^n(X; \pi_n(Y))\) is epic for all \(n\),
(b) \((i \wedge i)^* : H^n(F \wedge F; \text{coker } f_n) \to H^n(\Omega Y \wedge \Omega Y; \text{coker } f_n)\) is monic for all \(n\) (where \(f_n : \pi_n(X) \to \pi_n(Y)\)).

If \(X\) is an \(H\)-space, then \(F\) is an \(H\)-space such that \(\pi : F \to X\) is an \(H\)-map.

By adding some stability criteria to Theorem 0.1 we prove

**Theorem 0.2.** Assume \(Y\) is \((p - 1)\)-connected, \(F\) is \((q - 1)\)-connected \((p - 1 \geq 2, q - 1 \geq 1)\) of dimension \(< \min(2p - 1, p + q - 1)\), and \(H^*(Y)\) is free. If \(X\) is an \(H\)-space and \(f^* : H^n(Y) \to H^n(X)\) is onto for all \(n\) then \(F\)
is an H-space and the map \( \pi : F \to X \) is an H-map. (\( H^*(\cdot) \) is cohomology with integer coefficients.)

**Theorem 0.3.** Let \( X \) be a loop space \((X(n-1)-\)connected). If the induced maps \( \ell_q : \pi_q(X) \to \pi_q(Y) \) and \( f^* : H^q(Y; \pi_q(Y)) \to H^q(X; \pi_q(Y)) \) are onto for all \( q \) and all \( q > n \) respectively, then \( F \) is a loop space and the map \( \pi : F \to X \) is a loop map.

**Theorem 0.4.** Assume \( Y \) is \((p-1)-\)connected, \( F \) is \((q-1)-\)connected \((p-1 \geq 2, q-1 \geq 1) \) of dimension \(< \min(2p-1, p+q-1) \), and \( H(Y) \) is free. If \( X \) is a loop space and \( f^* : H^n(Y) \to H^n(X) \) is onto for all \( n \) then \( F \) is a loop space and \( \pi : F \to X \) is a loop map.

Let \((X, m)\) and \((Y, n)\) be H-spaces. Following Stasheff \[2\], given \( f : X \to Y \) an H-map (i.e., \( n(f \times f) \) is homotopic to \( fm \)), we define \( X \) an \( f\)-sub-H-space of \( Y \) if \( f \) is of the homotopy type of the inclusion of the fibre \( F \) into the total space \( E \) of a fibration over a space \( B \).

It is easy to see that this is equivalent to the following:

Replace \( f : X \to Y \) by the fibration \( F_f \xrightarrow{i} P_f \xrightarrow{p} Y \), with \( X \simeq P_f \) and \( p \) equivalent to \( f \).

Then \( P_f \to Y \) is induced by a map \( Y \to B \).

If \( X \) and \( F_f \) are loop spaces and \( \pi : F_f \to X \) is a loop map, we can extend the fibration sequence \( F_f \to X = Y \) to \( F_f \to X \to Y \to B_F \to B_X \) so that as a corollary of Theorems 0.3 or 0.4 we get

**Theorem 0.5.** Let \( f : (X, m) \to (Y, n) \) be an H-map with \( X \) a loop space and \( m \) the loop multiplication. If the conditions of Theorems 0.3 or 0.4 are satisfied, then \( X \) is an \( f\)-sub-H-space of \( Y \).

In §1 we define and develop a \( \phi \)-Postnikov system and prove the general results needed for this paper. §2 contains a proof of a proposition which give conditions which are internal to the \( \phi \)-Postnikov construction that insures \( F \) is an H-space. In §3, we prove a theorem which immediately specializes to Theorem 0.1, and in §4, we prove Theorem 0.2. In §5 we will make the obvious generalization to loop spaces. I would like to thank Professor Arthur Copeland, Jr. for his suggestions and comments.

1. \( \phi \)-Postnikov systems. Let \( \phi : \mathbb{Z}^+ \to \mathbb{Z}^+ \) be a nondecreasing function from the nonnegative integers onto themselves and let \( X \) be a connected simple CW complex with \( \pi_1(X) \) abelian. We define a \( \phi \)-Postnikov system for \( X \) as follows:

**Definition 1.1.** A \( \phi \)-Postnikov system \( \{X_n, \pi_n, p_n, q_n, k^n\} \) for \( X \) consists
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of a family of spaces $X_n (X_0 = \ast)$ and abelian groups $\pi_n, n \in \mathbb{Z}^+$ with maps $p_n : X_n \to X_{n-1}, q_n : X \to X_n,$ and $k^n : X_n \to K(\pi_n, \phi(n) + 2)$ such that

1. $q_{n-1} = p_n \circ q_n,$
2. $p_n : X_n \to X_{n-1}$ is equivalent to the fibration induced by $k^n,$
3. $(q_n)_* : \pi_i(X) \to \pi_i(X_n)$ is an isomorphism for $i \leq \phi(n); \pi_i(X_n) = 0,$ $i > \phi(n) + 1.$

The following are examples of $\phi$-Postnikov systems.

Example 1.2. Let $\{X_n, \pi_n, p_n, q_n, k^n\}$ be the standard Postnikov system for $X,$ i.e., $\phi = \text{Id}, \pi_n = 0(x) \text{ and } k^n \text{ a representative of the transgression of the fundamental class of the fibre of } q_{n-1} : X \to X_{n-1}$ (e.g., [1] or [4]).

Example 1.3. Let $X$ be a space and let $f_n : \pi_n(x) \to \pi'_n$ be an epimorphism for each $n.$ Let $\phi(n) = \lfloor n/2 \rfloor$ (the greatest integer less than or equal to $n$) and let $f_{nc} : K(\pi_{n+1}(x), n + 2) \to K(\pi_{n+1}, n + 2)$ be a realization of $f_n.$ We then may construct the $\phi$-Postnikov system $\{X_n, \pi_n, p_n, q_n, k^n\}$ (with $\pi_{2n} = \pi'_{n+1}, \pi_{2n+1} = \ker f_n$) inductively as follows:

Assume we have constructed

$X_{2n} \xrightarrow{p_{2n}} X_{2n-1} \to \cdots \to X_0 = \ast$ with $q_{2n} : X \to X_{2n}$

inducing isomorphisms in homotopy in dimensions $\leq \phi(2n) = n$ and $\pi_i(X_{2n}) = 0$ for $i > n.$ Let $k^{2n} : X_{2n} \to K(\pi'_n, n + 2)$ be the composition $f_{(n+1)c} \circ k^n,$ where $k^n$ is constructed as in Example 1.2, and let $X_{2n+1}$ be the total space of the fibration induced by $k^{2n}.$ It is easy to see that $q_{2n+1}$ and $p_{2n+1}$ can be constructed satisfying the conditions of Definition 1.1.

If we let $F_n$ be the fibre of $q_{2n+1} : X \to X_{2n+1}$ we see that $\pi_{n+1}(F_n) = \ker f_n.$ Let $k^{2n+1} : X_{2n+1} \to K(\pi_{n+1}(F_n), n + 2)$ be a representative of the transgression of the fundamental class. $X_{2n+2}$ may now be constructed with the desired properties.

Example 1.4. Let $f : X \to Y.$ Then by a suitable refinement of Example 1.3 we may construct $\phi$-Postnikov systems $\{X_n, \pi_n^X, p_n^X, q_n^X, k_n^X\}$ and $\{Y_n, \pi_n^Y, p_n^Y, q_n^Y, k_n^Y\}$ for $X$ and $Y$ respectively, such that $\pi_{2n}^X = \im f_{n+1},$ $\pi_{2n+1}^X = \ker f_{n+1}, \pi_n^Y = \pi_{n+1}(Y) \text{ and } \pi_{2n+1}^Y = 0.$

Further, we may construct these spaces and maps (see [1]) to yield the following commutative diagrams:

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$f_m : X_m \to Y_m$ is induced from $f_{m-1}$ by considering $X_m$ and $Y_m$ as the total spaces of the fibrations induced by $k_{X}^{m-1}$ and $k_{Y}^{m-1}$ respectively, with the first nontrivial $f_m$ the realization of the coefficient homomorphism $f_\ast$ in homotopy.

In much of what follows, we will need homotopy commutativity of

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow f_m & & \downarrow q_m \\
X_m & \xrightarrow{f_m} & Y_m
\end{array}
\]

To do this, we need the following lemma.

**Lemma 1.6.** If $X$ is $(r-1)$-connected and $Y$ is $(s-1)$-connected and $f^\ast : H^q(Y; \pi_q(Y)) \to H^q(X; \pi_q(Y))$ is onto for $q > \max(r+1, s+1)$, then there exist $\phi$-Postnikov systems for $X$ and $Y$ such that (1.5) homotopy commutes.

**Proof.** We use an inductive argument. Assume $f_m : q_m^X \sim q_m^Y$. If $m = 2n+1$, $Y_{2n+2} = Y_{2n+1}$ and $f_{2n+2}$ may be defined by $f_{2n+1} : q_m^X \to q_m^Y$.

If $m = 2n$ we have that $Y_{2n+1} \to Y_{2n}$ is a principal fibration of type $(\pi_{n+1}(Y), n+2)$. Since $p_Y f_{2n+1} q_m^X \sim p_Y q_m^Y$, there is an element

\[\tau \in [X, K(\pi_{n+1}(Y), n+1)] = H^{n+1}(X; \pi_{n+1}(Y))\]

such that $\psi(r, q_m^Y) = f_{2n+1} q_m^X$ (see [4]), where $\psi$ is the action of the fibre of $Y_{2n+1} \to Y_{2n}$ on $Y_{2n+1}$.

If $f^\ast$ is onto take $\tau' \in [Y, K(\pi_{n+1}(Y), n+1)] = H^{n+1}(Y; \pi_{n+1}(Y))$ in $f^{-1}(r)$ and change the lifting $q_m^Y$ of $q_m^X$ to the lifting $\psi(\tau', q_m^Y) = q_m^X$. By the naturality of the action $\psi$ we get the required result.

In all that follows we will assume that $f : X \to Y$ satisfies the hypothesis of Lemma 1.6 and call that condition Obl.

If we define $F_m$ as the total space of the fibration induced by $f_m$ we get
a $\phi$-Postnikov system for $F (\phi = [n - 1/2])$, $\{F_n, \pi_n^F, p_n^F, q_n^F, k_n^F\}$ with $\pi_n^F = \text{coker } f_{n+1*}, \pi_{2n+1}^F = \text{ker } f_{n+1*}$ and such that the following diagrams commute:

![Diagram](image)

(1.6)

The maps $\pi$ and $p_m^F : F_m \to F_{m-1}$ are determined by considering $F_m$ as the total space of the fibration induced by $k_m^F$.

If $X$ is an $H$-space ($\Omega$ space), then the spaces $X_m$ are $H$-spaces ($\Omega$-spaces) and the maps $q_m^X$, $p_m^X$, and $k_m^X$ are $H$-maps. If each $F_m$ is an $H$-space ($\Omega$ space) and each $k_m^F$ is an $H$-map ($\Omega$-map) then $F$ is an $H$-space ($\Omega$-space), and if the maps $\pi_m : F_m \to X_m$ are $H$-maps ($\Omega$-maps) then so is $\pi : F \to X$ (see Kahn [1], Stasheff [2], [3]).

2. Internal techniques. To determine conditions under which $F$ is an $H$-space the following two technical lemmas are useful.

**Lemma 2.1.** If $F_{2n-1}$ and $X_{2n-1}$ are $H$-spaces and the maps $\pi_{2n-1}$ and $k_{2n-1}^X$ are $H$-maps then $F_{2n}$ and $X_{2n}$ are $H$-spaces and the maps $\pi_{2n}$ and $k_{2n}^F$ are $H$-maps.
Lemma 2.2. If $F_{2n}$ and $X_{2n}$ are $H$-spaces and the maps $\pi_{2n}, k_{X}^{2n}$ and $k_{F}^{2n}$ are H-maps then $F_{2n+1}$ and $X_{2n+1}$ are $H$-spaces and the map $\pi_{2n+1}$ is an $H$-map. (Note that (1.6) commutes so that the maps $k_{X}^{2n}$ and $k_{F}^{2n}$ are compatible $H$-maps.)

The proofs of these lemmas are a direct application of Theorem 2 of [2] in diagrams (1.6) and (1.7).

Since in the stable range Lemmas 2.1 and 2.2 hold, these two lemmas can then be used inductively to get

Proposition 2.3. Let $f : X \to Y$ satisfy condition Obl and let $X$ be an $H$-space. If $k_{F}^{2n} : F_{2n} \to K(\coker f_{n+1*}, n+1)$ is an $H$-map for all $n$ then $F$ is an $H$-space and the map $\pi : F \to X$ is an $H$-map.

The obstruction to

$$k_{F}^{2n} : F_{2n} \to K(\coker f_{n+1*}, n+1)$$

being an $H$-map lies in $H^{n+1}(F_{2n} \wedge F_{2n}; \coker f_{n+1*})$.

If we consider the diagram

$$\begin{array}{ccc}
\Omega Y_{2n} & \xrightarrow{\Omega k_{Y}^{2n}} & K(\pi_{n+1}(\Omega Y), n+1) \\
i_{2n} & & j_{2n} \\
F_{2n} & \xrightarrow{f_{2n}} & K(\coker f_{n+1*}, n+1)
\end{array}$$

induced from Example 1.3 we have that $k_{F}^{2n} \circ i_{2n} = i_{2n} \circ \Omega k_{Y}^{2n}$ and that $i_{2n}$, $\Omega k_{Y}^{2n}$ and $j_{2n}$ are all $H$-maps.

Therefore, if we let $\sigma(H) \in H^{n+1}(F_{2n} \wedge F_{2n}; \coker f_{n+1*})$ be the obstruction to $k_{F}^{2n}$ being an $H$-map we get $(i_{2n} \wedge i_{2n})^{*}(\sigma(H))$ is zero in $H^{n+1}(\Omega Y_{2n} \wedge \Omega Y_{2n}; \coker f_{n+1*})$.

In particular we get

Proposition 2.4. Let $f : X \to Y$ satisfy condition Obl and let $X$ be an $H$-space. If the map

$$(i_{2n} \wedge i_{2n})^{*} : H^{n+1}(F_{2n} \wedge F_{2n}; \coker f_{n+1*}) \to H^{n+1}(\Omega Y_{2n} \wedge \Omega Y_{2n}; \coker f_{n+1*})$$

is monic for all $n$ then $F$ is an $H$-space and the map $\pi : F \to X$ is an $H$-map.
Theorem 3.1. Let $f: X \to Y$ satisfy Obi. Let $F$, $\Omega Y$ be simply connected and let $X$ be an H-space. If the map

$$(i \wedge i)^* : H^{n+1}(F \wedge F; \text{coker } f_{n+1}^*) \to H^{n+1}(\Omega Y \wedge \Omega Y; \text{coker } f_{n+1}^*)$$

is monic for each $n$; then $F$ is an H-space and the map $\pi : F \to X$ is an H-map.

Note that this theorem requires conditions on $f: X \to Y$ and not on the stages of any $\phi$-Postnikov system. Further note that Theorem 0.1 is an immediate corollary of this.

Proof of Theorem 3.1. By the universal coefficient theorem

$$H^{n+1}(K \wedge K; \text{coker } f_{n+1}^*)$$

$$\cong \text{Hom}(H_{n+1}(K \wedge K); \text{coker } f_{n+1}^*) \oplus \text{Ext}(H_n(K \wedge K); \text{coker } f_{n+1}^*)$$

for $K = F_{2n}$ or $\Omega Y_{2n}$.

By the Künneth formula

$$H_m(K \wedge K) \cong \sum_{p+q=m; m>0} H_p(K) \otimes H_q(K) \oplus \sum_{p+q=m-1} \text{Tor}(H_p(K), H_q(K)).$$

But $q_{2n}^F : F \to F_{2n}$ and $q_{2n}^Y : \Omega Y \to \Omega Y_{2n}$ are both $n-1$ equivalences so that letting $K = F_{2n}$ (or $\Omega Y_{2n}$) and $L = F$ (or $\Omega Y$) we get

$$H_{n+1}(K \wedge K) \cong (H_n(K) \otimes H_1(K))^2 \oplus \sum_{p+q=n+1} H_p(L) \otimes H_q(L)$$

$$\oplus \sum_{p+q=n} \text{Tor}(H_p(L), H_q(L)).$$

Since $K$ is simply connected we get that $q : L \to K$ induces isomorphisms $(q \wedge q)_* : H_i(L \wedge L) \to H_i(K \wedge K)$ for $i \leq n+1$ and, therefore, we may replace the condition of Proposition 2.4 that the map $(i_{2n} \wedge i_{2n})^*$ be monic with the condition that $(i \wedge i)^*$ be monic.

Proof of Theorem 0.2. We will now assume that $H_*(Y)$ is free and that $Y$ is $(p-1)$-connected and $F$ is $(q-1)$-connected. From the Serre spectral sequence for the "fibration" $F \to X \to Y$ we get that

$$\cdots \to H^{i-1}(F) \xrightarrow{\delta^F} H^i(Y) \xrightarrow{f^*} H^i(X) \xrightarrow{\pi^*} H^i(F)$$

is exact for $i \leq p + q - 1$.  

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Therefore, if \( i^*: H^i(Y) \to H^i(X) \) is epic for all \( i \) we get that \( H^{i-1}(\Omega Y) \Rightarrow Y \quad H^i(Y) \) is an isomorphism for \( i \leq 2p - 1 \). Thus we get \( i^*: H^i(F) \to H^i(\Omega Y) \) is monic for \( i < \min(2p - 1, p + q - 1) = l \).

If the dimension of \( F \) is less than or equal to \( l \) we get \( i^*: H^i(F) \to H^i(\Omega Y) \) is monic for all \( i \), and since \( H^i(Y) \) is free this implies \( (i \wedge i)^*: H^i(F \wedge F) \to H^i(\Omega Y \wedge \Omega Y) \) is monic. Thus Theorem 0.2 is proven by appealing to Theorem 0.1.

5. Loop spaces. Since 2.1, 2.2, and 2.3 are also trivially true if we replace \( H \)-space and \( H \)-map by (the homotopy type of a) loop space and loop map, and since the obstructions to \( k^{2^n}_F: F^{2n} \to K(\text{coker} f_{n+1*}, n + 1) \) being a loop map are elements

\[
\sigma^i_i(\Omega) \in H^{n+1}(\wedge^i(F^{2n}); \text{coker } f_{n+1*}) \quad (\wedge^i(F^{2n}) = F^{2n} \wedge \cdots \wedge F^{2n})
\]

it is easy to see that we have the following result.

**Proposition 5.1.** Let \( f: X \to Y \) satisfy condition 1.2 and let \( X \) be a loop space. If the maps

\[
(\wedge^i i^{2n}_2)^*: H^{n+1}(\wedge^i(F^{2n}); \text{coker } f_{n+1*}) \to H^{n+1}(\wedge^i(\Omega Y^{2n}); \text{coker } f_{n+1*})
\]

are monic for all \( n \) and all \( j \geq 2 \), then \( f \) is a loop space and the map \( \pi: F \to X \) is a loop map.

As in §3, if we take \( F \) and \( \Omega Y \) we get a result analogous to Theorem 3.1 but we need the condition \( (\wedge^i i)^* \) is monic for all \( n \) and \( j \geq 2 \), a condition too cumbersome to be of much value. By adding the condition, \( H^*(Y) \) is free we only need \( i^{2n}_2 \) monic, and hence we get Theorem 0.4. By assuming that \( \pi_q(X) \to \pi_q(Y) \) is onto, we get \( \text{coker } f_{n*} = 0 \), and hence Proposition 5.1 implies Theorem 0.3.

**REFERENCES**


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