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ϕ-POSTNIKOV SYSTEMS AND EXTENSIONS OF H-SPACES

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ABSTRACT. Let $f: X \to Y$ be a map of CW complexes and let $\pi: F_f \to X$ be the fibration induced by f.

The following theorems are proven:

Theorem. *Assume F $(=F_f)$ and ΩY are simply connected and that*

(a) $f^*: H^n(Y; \pi_n(Y)) \to H^n(X; \pi_n(Y))$ *is epic for all* n,

(b) $(i \wedge i)^*: H^n(F \wedge F; \operatorname{coker} f_{n*}) \to H^n(\Omega Y \wedge \Omega Y; \operatorname{coker} f_{n*})$ *is monic for all n (where $f_{n*}: \pi_n(X) \to \pi_n(Y)$).*

If X is an H-space then F is an H-space such that $\pi: F \to X$ is an H-map.

Theorem. *Assume Y is $(p-1)$-connected, F is $(q-1)$-connected $(p-1 \geq 2,\ q-1 \geq 1)$ of dimension $< \min(2p-1, p+q-1)$, and $H^*(Y)$ is free. If X is an H-space and $f^*: H^n(Y) \to H^n(X)$ is onto for all n then F is an H-space and the map $\pi: F \to X$ is an H-map.*

Analogous theorems are shown to hold for loop spaces.

Let $f: X \to Y$ be a (based) map of CW spaces and let $\Omega Y \xrightarrow{i} F_f \xrightarrow{\pi} X$ be the fibration induced by f. That is, $F_f = \{(x, \eta) \in X \times PY \mid f(x) = \eta(1)\}$ and $\pi(x, \eta) = x$.

In this paper we will develop a Postnikov-type decomposition which will allow us to prove

Theorem 0.1. *Assume F $(=F_f)$ and ΩY are simply connected and that*

(a) $f^*: H^n(Y; \pi_n(Y)) \to H^n(X; \pi_n(Y))$ *is epic for all* n,

(b) $(i \wedge i)^*: H^n(F \wedge F; \operatorname{coker} f_{n*}) \to H^n(\Omega Y \wedge \Omega Y; \operatorname{coker} f_{n*})$ *is monic for all n (where $f_{n*}: \pi_n(X) \to \pi_n(Y)$).*

If X is an H-space, then F is an H-space such that $\pi: F \to X$ is an H-map.

By adding some stability criteria to Theorem 0.1 we prove

Theorem 0.2. *Assume Y is $(p-1)$-connected, F is $(q-1)$-connected $(p-1 \geq 2,\ q-1 \geq 1)$ of dimension $< \min(2p-1, p+q-1)$, and $H^*(Y)$ is free. If X is an H-space and $f^*: H^n(Y) \to H^n(X)$ is onto for all n then F*

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is an H-space and the map $\pi: F \to X$ *is an H-map.* ($H^*(\ )$ *is cohomology with integer coefficients.*)

Theorem 0.3. *Let* X *be a loop space* (X $(n-1)$*-connected*). *If the induced maps* $f_*: \pi_q(X) \to \pi_q(Y)$ *and* $f^*: H^q(Y; \pi_q(Y)) \to H^q(X; \pi_q(Y))$ *are onto for all* q *and all* $q > n$ *respectively, then* F *is a loop space and the map* $\pi: F \to X$ *is a loop map.*

Theorem 0.4. *Assume* Y *is* $(p-1)$*-connected,* F *is* $(q-1)$*-connected* $(p-1 \geq 2,\ q-1 \geq 1)$ *of dimension* $< \min(2p-1, p+q-1)$, *and* $H^*(Y)$ *is free. If* X *is a loop space and* $f^*: H^n(Y) \to H^n(X)$ *is onto for all* n *then* F *is a loop space and* $\pi: F \to X$ *is a loop map.*

Let (X, m) and (Y, n) be H-spaces. Following Stasheff [2], given $f: X \to Y$ an H-map (i.e., $n(f \times f)$ is homotopic to fm), we define X an *f-sub-H-space* of Y if f is of the homotopy type of the inclusion of the fibre F into the total space E of a fibration over a space B.

It is easy to see that this is equivalent to the following:

Replace $f: X \to Y$ by the fibration $F_f \xrightarrow{i} P_f \xrightarrow{p} Y$, with $X \simeq P_f$ and p equivalent to f.

Then $P_f \to Y$ is induced by a map $Y \to B$.

If X and F_f are loop spaces and $\pi: F_f \to X$ is a loop map, we can extend the fibration sequence $F_f \to X \to Y$ to $F_f \to X \to Y \to B_F \to B_X$ so that as a corollary of Theorems 0.3 or 0.4 we get

Theorem 0.5. *Let* $f: (X, m) \to (Y, n)$ *be an H-map with* X *a loop space and* m *the loop multiplication. If the conditions of Theorems* 0.3 *or* 0.4 *are satisfied, then* X *is an f-sub-H-space of* Y.

In §1 we define and develop a ϕ-Postnikov system and prove the general results needed for this paper. §2 contains a proof of a proposition which give conditions which are internal to the ϕ-Postnikov construction that insures F is an H-space. In §3, we prove a theorem which immediately specializes to Theorem 0.1, and in §4, we prove Theorem 0.2. In §5 we will make the obvious generalization to loop spaces. I would like to thank Professor Arthur Copeland, Jr. for his suggestions and comments.

1. **ϕ-Postnikov systems.** Let $\phi: Z^+ \to Z^+$ be a nondecreasing function from the nonnegative integers onto themselves and let X be a connected simple CW complex with $\pi_1(X)$ abelian. We define a ϕ-Postnikov system for X as follows:

Definition 1.1. A ϕ-Postnikov system $\{X_n, \pi_n, p_n, q_n, k^n\}$ for X consists of a family of spaces X_n ($X_0 = *$) and abelian groups π_n, $n \in Z^+$ with maps $p_n : X_n \to X_{n-1}$, $q_n : X \to X_n$, and $k^n : X_n \to K(\pi_n, \phi(n) + 2)$ such that

(1) $q_{n-1} = p_n q_n$,

(2) $p_n : X_n \to X_{n-1}$ is equivalent to the fibration induced by k^n,

(3) $(q_n)_* : \pi_i(X) \to \pi_i(X_n)$ is an isomorphism for $i \leq \phi(n)$; $\pi_i(X_n) = 0$, $i > \phi(n) + 1$.

The following are examples of ϕ-Postnikov systems.

Example 1.2. Let $\{X_n, \pi_n, p_n, q_n, k^n\}$ be the standard Postnikov system for X, i.e., $\phi = \mathrm{Id}$, $\pi_n = \pi_n(x)$ and k^n a representative of the transgression of the fundamental class of the fibre of $q_{n-1} : X \to X_{n-1}$ (e.g., [1] or [4]).

Example 1.3. Let X be a space and let $f_n : \pi_n(x) \to \pi'_n$ be an epimorphism for each n. Let $\phi(n) = [n/2]$ ($[s]$, the greatest integer less than or equal to s) and let $f_{nc} : K(\pi_{n+1}(x), n + 2) \to K(\pi_{n+1}, n + 2)$ be a realization of f_n. We then may construct the ϕ-Postnikov system $\{X_n, \pi_n, p_n, q_n, k^n\}$ (with $\pi_{2n} = \pi'_{n+1}$, $\pi_{2n+1} = \ker f_n$) inductively as follows:

Assume we have constructed

$$X_{2n} \xrightarrow{p_{2n}} X_{2n-1} \to \cdots \to X_0 = * \qquad \text{with } q_{2n} : X \to X_{2n}$$

inducing isomorphisms in homotopy in dimensions $\leq \phi(2n) = n$ and $\pi_i(x_{2n}) = 0$ for $i > n$. Let $k^{2n} : X_{2n} \to K(\pi'_{n+1}, n + 2)$ be the composition $f_{(n+1)i} \circ k'^n$, where k'^n is constructed as in Example 1.2, and let X_{2n+1} be the total space of the fibration induced by k^{2n}. It is easy to see that q_{2n+1} and p_{2n+1} can be constructed satisfying the conditions of Definition 1.1.

If we let F_n be the fibre of $q_{2n+1} : X \to X_{2n+1}$ we see that $\pi_{n+1}(F_n) = \ker f_n$. Let $k^{2n+1} : X_{2n+1} \to K(\pi_{n+1}(F_n), n + 2)$ be a representative of the transgression of the fundamental class. X_{2n+2} may now be constructed with the desired properties.

Example 1.4. Let $f : X \to Y$. Then by a suitable refinement of Example 1.3 we may construct ϕ-Postnikov systems $\{X_n, \pi_n^X, p_n^X, q_n^X, k_X^n\}$ and $\{Y_n, \pi_n^Y, p_n^Y, q_n^Y, k_Y^n\}$ for X and Y respectively, such that $\pi_{2n}^X = \operatorname{im} f_{n+1*}$, $\pi_{2n+1}^X = \ker f_{n+1*}$ ($f_{n*} : \pi_n(X) \to \pi_n(Y)$), $\pi_{2n}^Y = \pi_{n+1}(Y)$ and $\pi_{2n+1}^Y = 0$.

Further, we may construct these spaces and maps (see [1]) to yield the following commutative diagrams:

$$\begin{array}{ccc} X_{2n} & \xrightarrow{k_X^{2n}} & K(\operatorname{im} f_{n+1*}, n + 2) \\ {\scriptstyle f_{2n}}\downarrow & & \downarrow \\ Y_{2n} & \longrightarrow & K(\pi_{n+1}(Y), n + 2) \end{array}$$

$$\begin{array}{ccc} X_{2n+1} & \longrightarrow & K(\ker f_{n+1*},\, n+2) \\ {\scriptstyle f_{2n+1}}\downarrow & & \downarrow \\ Y_{2n+1} & \longrightarrow & * \end{array}$$

$f_m : X_m \to Y_m$ is induced from f_{m-1} by considering X_m and Y_m as the total spaces of the fibrations induced by k_X^{m-1} and k_Y^{m-1} respectively, with the first nontrivial f_m the realization of the coefficient homomorphism f_* in homotopy.

In much of what follows, we will need homotopy commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle q_m^X}\downarrow & & \downarrow{\scriptstyle q_m^Y} \\ X_m & \xrightarrow[\ f_m\ ]{} & Y_m \end{array} \tag{1.5}$$

To do this, we need the following lemma.

Lemma 1.6. *If X is $(r-1)$-connected and Y is $(s-1)$-connected and $f^*: H^q(Y; \pi_q(Y)) \to H^q(X; \pi_q(Y))$ is onto for $q > \max(r+1, s+1)$, then there exist ϕ-Postnikov systems for X and Y such that* (1.5) *homotopy commutes.*

Proof. We use an inductive argument. Assume $f_m q_m^X \sim q_m^Y f$. If $m = 2n+1$, $Y_{2n+2} = Y_{2n+1}$ and f_{2n+2} may be defined by $f_{2n+1} p_{2n+2}^X q_{2n+2}^X$.

If $m = 2n$ we have that $Y_{2n+1} \to Y_{2n}$ is a principal fibration of type $(\pi_{n+1}(Y), n+2)$. Since $p^Y f_{2n+1} q_{2n+1}^X \sim p^Y q_{2n+1}^Y f$, there is an element

$$\tau \in [X,\ K(\pi_{n+1}(Y),\ n+1)] = H^{n+1}(X; \pi_{n+1}(Y))$$

such that $\psi(\tau, q_{2n+1}^Y f) = f_{2n+1} q_{2n+1}^X$ (see [4]), where ψ is the action of the fibre of $Y_{2n+1} \to Y_{2n}$ on Y_{2n+1}.

If f^* is onto take $\tau' \in [Y; K(\pi_{n+1}(Y), n+1)] = H^{n+1}(Y; \pi_{n+1}(Y))$ in $f^{*-1}(\tau)$ and change the lifting q_{2n+1}^Y of q_{2n}^Y to the lifting $\psi(\tau', q_{2n+1}^Y) = q'^Y_{2n+1}$. By the naturality of the action ψ we get the required result.

In all that follows we will assume that $f: X \to Y$ satisfies the hypothesis of Lemma 1.6 and call that condition Obl.

If we define F_m as the total space of the fibration induced by f_m we get a ϕ-Postnikov system for F ($\phi = [n - 1/2]$), $\{F_n, \pi_n^F, p_n^F, q_n^F, k_F^n\}$ with $\pi_{2n}^F =$ coker f_{n+1*}, $\pi_{2n+1}^F = \ker f_{n+1*}$ and such that the following diagrams commute:

$$
(1.6)\qquad
\begin{array}{ccc}
F_{2n} & \xrightarrow{\;k_F^{2n}\;} & K(\operatorname{coker} f_{n+1*},\ n+1) \\
\Big\downarrow{\scriptstyle \pi_{2n}} & & \Big\downarrow{\scriptstyle \Delta_{2n}} \\
X_{2n} & \xrightarrow{\;k_X^{2n}\;} & K(\operatorname{im} f_{n+1*},\ n+2) \\
\Big\downarrow{\scriptstyle f_{2n}} & & \Big\downarrow \\
Y_{2n} & \longrightarrow & K(\pi_{n+1}(Y),\ n+2)
\end{array}
$$

$$
(1.7)\qquad
\begin{array}{ccc}
F_{2n+1} & \xrightarrow{\;k_F^{2n+1}\;} & K(\ker f_{n+1*},\ n+2) \\
\Big\downarrow{\scriptstyle \pi_{2n+1}} & & \Big\downarrow \\
X_{2n+1} & \xrightarrow{\;k_X^{2n+1}\;} & K(\ker f_{n+1*},\ n+2) \\
\Big\downarrow{\scriptstyle f_{2n+1}} & & \Big\downarrow \\
Y_{2n+1} & \longrightarrow & *
\end{array}
$$

The maps π and $p_m^F : F_m \to F_{m-1}$ are determined by considering F_m as the total space of the fibration induced by k_F^{m-1}.

If X is an H-space (Ω space), then the spaces X_m are H-spaces (Ω-spaces) and the maps q_m^X, p_m^X, and k_X^m are H-maps. If each F_m is an H-space (Ω space) and each k_F^m is an H-map (Ω-map) then F is an H-space (Ω-space), and if the maps $\pi_m : F_m \to X_m$ are H-maps (Ω-maps) then so is $\pi : F \to X$ (see Kahn [1], Stasheff [2], [3]).

2. **Internal techniques.** To determine conditions under which F is an H-space the following two technical lemmas are useful.

Lemma 2.1. *If F_{2n-1} and X_{2n-1} are H-spaces and the maps π_{2n-1} and k_X^{2n-1} are H-maps then F_{2n} and X_{2n} are H-spaces and the maps π_{2n} and k_F^{2n-1} are H-maps.*

Lemma 2.2. *If F_{2n} and X_{2n} are H-spaces and the maps π_{2n}, k_X^{2n} and k_F^{2n} are H-maps then F_{2n+1} and X_{2n+1} are H-spaces and the map π_{2n+1} is an H-map. (Note that (1.6) commutes so that the maps k_X^{2n} and k_F^{2n} are compatible H-maps.)*

The proofs of these lemmas are a direct application of Theorem 2 of [2] in diagrams (1.6) and (1.7).

Since in the stable range Lemmas 2.1 and 2.2 hold, these two lemmas can then be used inductively to get

Proposition 2.3. *Let $f: X \longrightarrow Y$ satisfy condition Obl and let X be an H-space. If $k_F^{2n}: F_{2n} \longrightarrow K(\mathrm{coker}; f_{n+1*}, n+1)$ is an H-map for all n then F is an H-space and the map $\pi: F \longrightarrow X$ is an H-map.*

The obstruction to

$$k_F^{2n}: F_{2n} \longrightarrow K(\mathrm{coker}; f_{n+1*}, n+1)$$

being an H-map lies in $H^{n+1}(F_{2n} \wedge F_{2n}; \mathrm{coker}\, f_{n+1*})$.

If we consider the diagram

$$\begin{array}{ccc} \Omega Y_{2n} & \xrightarrow{\Omega k_Y^{2n}} & K(\pi_{n+1}(\Omega Y), n+1) \\ \big\downarrow{\scriptstyle i_{2n}} & & \big\downarrow{\scriptstyle j_{2n}} \\ F_{2n} & \longrightarrow & K(\mathrm{coker}; f_{n+1*}, n+1) \end{array}$$

induced from Example 1.3 we have that $k_F^{2n} \circ i_{2n} = j_{2n} \circ \Omega k_Y^{2n}$ and that i_{2n}, Ωk_Y^{2n} and j_{2n} are all H-maps.

Therefore, if we let $\sigma(H) \in H^{n+1}(F_{2n} \wedge F_{2n}; \mathrm{coker}\, f_{n+1*})$ be the obstruction to k_F^{2n} being an H-map we get $(i_{2n} \wedge i_{2n})^*(\sigma(H))$ is zero in $H^{n+1}(\Omega Y_{2n} \wedge \Omega Y_{2n}; \mathrm{coker}\, f_{n+1*})$.

In particular we get

Proposition 2.4. *Let $f: X \longrightarrow Y$ satisfy condition Obl and let X be an H-space. If the map*

$$(i_{2n} \wedge i_{2n})^*: H^{n+1}(F_{2n} \wedge F_{2n}; \mathrm{coker}\, f_{n+1*}) \longrightarrow H^{n+1}(\Omega Y_{2n} \wedge \Omega Y_{2n}; \mathrm{coker}\, f_{n+1*})$$

is monic for all n then F is an H-space and the map $\pi: F \longrightarrow X$ is an H-map.

3. **External conditions and a proof of Theorem 0.1.** Using 2.4 we may prove

Theorem 3.1. *Let* $f: X \to Y$ *satisfy Obl. Let* F, ΩY *be simply connected and let* X *be an H-space. If the map*

$$(i \wedge i)^*: H^{n+1}(F \wedge F; \operatorname{coker} f_{n+1*}) \to H^{n+1}(\Omega Y \wedge \Omega Y; \operatorname{coker} f_{n+1*})$$

is monic for each n*; then* F *is an H-space and the map* $\pi: F \to X$ *is an H-map.*

Note that this theorem requires conditions on $f: X \to Y$ and not on the stages of any ϕ-Postnikov system. Further note that Theorem 0.1 is an immediate corollary of this.

Proof of Theorem 3.1. By the universal coefficient theorem

$$H^{n+1}(K \wedge K; \operatorname{coker} f_{n+1*})$$
$$\simeq \operatorname{Hom}(H_{n+1}(K \wedge K); \operatorname{coker} f_{n+1*}) \oplus \operatorname{Ext}(H_n(K \wedge K); \operatorname{coker} f_{n+1*})$$

for $K = F_{2n}$ or ΩY_{2n}.

By the Künneth formula

$$H_m(K \wedge K) \cong \sum_{p+q=m;\, m>p>0} H_p(K) \otimes H_q(K) \oplus \sum_{p+q=m-1} \operatorname{Tor}(H_p(K), H_q(K)).$$

But $q^F_{2n}: F \to F_{2n}$ and $q^Y_{2n}: \Omega Y \to \Omega Y_{2n}$ are both $n-1$ equivalences so that letting $K = F_{2n}$ (or ΩY_{2n}) and $L = F$ (or ΩY) we get

$$H_{n+1}(K \wedge K) \cong (H_n(K) \otimes H_1(K))^2 \oplus \sum_{p+q=n+1} H_p(L) \otimes H_q(L)$$
$$\oplus \sum_{p+q=n} \operatorname{Tor}(H_p(L), H_q(L)).$$

Since K is simply connected we get that $q: L \to K$ induces isomorphisms $(q \wedge q)_*: H_i(L \wedge L) \to H_i(K \wedge K)$ for $i \leq n+1$ and, therefore, we may replace the condition of Proposition 2.4 that the map $(i_{2n} \wedge i_{2n})^*$ be monic with the condition that $(i \wedge i)^*$ be monic.

Proof of Theorem 0.2. We will now assume that $H_*(Y)$ is free and that Y is $(p-1)$-connected and F is $(q-1)$-connected. From the Serre spectral sequence for the "fibration" $F \to X \to Y$ we get that

$$\cdots \to H^{i-1}(F) \xrightarrow{\delta_F} H^i(Y) \xrightarrow{f^*} H^i(X) \xrightarrow{\pi^*} H^i(F)$$

is exact for $i \leq p+q-1$.

Therefore, if $f^*: H^i(Y) \to H^i(X)$ is epic for all i we get that $H^{i-1}(\Omega Y) \xrightarrow{\delta_Y} H^i(Y)$ is an isomorphism for $i \leq 2p - 1$. Thus we get $i^*: H^i(F) \to H^i(\Omega Y)$ is monic for $i < \min(2p - 1, p + q - 1) = l$.

If the dimension of F is less than or equal to l we get $i^*: H^i(F) \to H^i(\Omega Y)$ is monic for all i, and since $H^i(Y)$ is free this implies $(i \wedge i)^*: H^i(F \wedge F) \to H^i(\Omega Y \wedge \Omega Y)$ is monic. Thus Theorem 0.2 is proven by appealing to Theorem 0.1.

5. **Loop spaces.** Since 2.1, 2.2, and 2.3 are also trivially true if we replace H-space and H-map by (the homotopy type of a) loop space and loop map, and since the obstructions to $k_F^{2n}: F_{2n} \to K(\text{coker}; f_{n+1*}, n + 1)$ being a loop map are elements

$$\sigma_i(\Omega) \in H^{n+1}(\wedge^i(F_{2n}); \text{coker } f_{n+1*}) \quad (\wedge^i(F_{2n}) = F_{2n} \wedge \cdots \wedge F_{2n})$$

it is easy to see that we have the following result.

Proposition 5.1. *Let $f: X \to Y$ satisfy condition* 1.2 *and let X be a loop space. If the maps*

$$(\wedge^j i_{2n})^*: H^{n+1}(\wedge^j(F_{2n}); \text{coker } f_{n+1*}) \to H^{n+1}(\wedge^j(\Omega Y_{2n}); \text{coker } f_{n+1*})$$

are monic for all n and all $j \geq 2$, then f is a loop space and the map π: $F \to X$ is a loop map.

As in §3, if we take F and ΩY we get a result analogous to Theorem 3.1 but we need the condition $(\wedge^j i)^*$ is monic for all n and $j \geq 2$, a condition too cumbersome to be of much value. By adding the condition, $H^*(Y)$ is free we only need i_{2n} monic, and hence we get Theorem 0.4. By assuming that $\pi_q(X) \to \pi_q(Y)$ is onto, we get coker $f_{n*} = 0$, and hence Proposition 5.1 implies Theorem 0.3.

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