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SUMS OF STATIONARY SEQUENCES CANNOT GROW SLOWER THAN LINEARLY

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ABSTRACT. It is shown that for a stationary sequence of random variables $X_1, X_2, \cdots$ one has

$$\liminf n^{-1} \sum_{i=1}^{n} X_i > 0$$

a.e. on the set $\{\Sigma_1^n X_i \to \infty, n \to \infty\}$.

1. **Introduction.** Let $(\Omega, \mathcal{B}, P)$ be a probability space and $X_1, X_2, \cdots$ a stationary sequence of $\mathcal{B}$-measurable functions, i.e. for Borel sets $B_1, B_2, \cdots$ of the real line

$$P\{X_1 \in B_1, \cdots, X_m \in B_m\} = P\{X_k \in B_1, X_{k+1} \in B_2, \cdots, X_{m+k-1} \in B_m\}, \qquad k \geq 1;$$

see [1, §6.1] for more details.[1] Birkhoff's ergodic theorem (see [1, Theorem 6.21, §6.6]) states that for $S_n = \Sigma_1^n X_i$

$$\lim_{n\to\infty} \frac{1}{n} S_n \text{ exists almost everywhere,} \tag{1.1}$$

whenever

$$\int_\Omega X_1 dP \tag{1.2}$$

is well defined. ((1.2) is well defined as long as at least one of

$$\int_\Omega X_1^+ dP \quad \text{and} \quad \int_\Omega X_1^- dP \tag{1.3}$$

are finite. In this case (1.2) is the difference of the two integrals in (1.3).)

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[1]Some people may prefer to use the following equivalent description: Let $(\Omega', \mathcal{B}', P')$ be a probability space, T a measure preserving transformation on Ω (i.e. $P\{T^{-1}A\} = P\{A\}$, $A \in \mathcal{B}'$) and $f: \Omega' \to \mathbf{R}$ a $\mathcal{B}'$-measurable function. Then $X_i'(\omega') = f(T^i\omega')$, $i = 1, 2, \cdots, \omega' \in \Omega'$ is a stationary sequence and the S_n below become $\Sigma_1^n f(T^i\omega')$.

The ergodic theorem also identifies the limit in (1.1), but our only concern right now is with those sequences for which the limit in (1.1) equals zero. For such sequences we merely obtain that $|S_n| = o(n)$, but it is conceivable that S_n goes off to $+\infty$ or $-\infty$ at a certain rate slower than linearly, e.g. that

$$S_n/n \to 0 \quad \text{but} \quad \liminf S_n/n^{1/2} > 0.$$

We shall show here that this is impossible; $|S_n|$ has to grow at least linearly when S_n goes to $+\infty$ or $-\infty$ *and this holds even when* (1.2) *is not defined.* More precisely we prove

Theorem. *If* $X_1, X_2, \cdots$ *is a stationary sequence, then*

$$\liminf_{n\to\infty} \frac{1}{n} S_n > 0 \text{ a.e. on } \{S_n \to \infty\}.$$

The proof is a simple application of the stationarity of $X_1, X_2, \cdots$ and the ergodic theorem. It uses a trick which was exploited by Wolfowitz [3] to prove the Poincaré recurrence theorem (see also [1, §§6.9, 6.10]).

Proof of theorem. Let $\Omega_0 = \{S_n \to \infty\}$. If $P\{\Omega_0\} = 0$ there is nothing to prove, and if $P\{\Omega_0\} > 0$ we may assume without loss of generality that Ω_0 is all of Ω. Indeed, we can then replace $(\Omega, \mathcal{B}, P)$ by $(\Omega_0, \mathcal{B}_0, P_0)$, where $\mathcal{B}_0$ is the trace of $\mathcal{B}$ on Ω_0 and $P_0(A) = P\{A\}/P\{\Omega_0\}$ for $A \in \mathcal{B}_0$. One easily checks that the restrictions of $X_1, X_2, \cdots$ to Ω_0 now form a stationary sequence on $(\Omega_0, \mathcal{B}_0, P_0)$. For the remainder we drop the subscript zero and assume that $S_n \to \infty$ everywhere. By a standard construction (see [1, Proposition 6.5]), we may assume moreover, that there exist also $\mathcal{B}_0$-measurable functions $X_0, X_{-1}, X_{-2}, \cdots$ on Ω such that the full sequence $\{X_i\}_{-\infty<i<\infty}$ is stationary. Now define $S_0 = 0$ and

$$\nu_0 = \min\left\{j \geq 0: \inf_{n>j} S_n - S_j > 0\right\} = \min\left\{j \geq 0: \inf_{n>j} \sum_{i=j+1}^{n} X_i > 0\right\}$$

$$= \text{last index at which } \min_{n\geq 0} S_n \text{ is achieved.}$$

Since $S_n \to \infty$ it is clear that $\nu_0 < \infty$, and that we can define successively

$$\nu_{k+1} = \min\left\{j > \nu_k: \inf_{n>j} S_n - S_j > 0\right\}, \tag{2.1}$$

and that all $\nu_k < \infty$. Next we observe that by stationarity

$$P\{\nu_0 = l\} \le P\left\{\inf_{n>l} S_n - S_l > 0\right\} = P\left\{\inf_{n>l} \sum_{i=l+1}^{n} X_i > 0\right\}$$

$$= P\left\{\inf_{n>0} S_n - S_0 > 0\right\} = P\{\nu_0 = 0\},$$

so that

$$1 = \sum_{l=0}^{\infty} P\{\nu_0 = l\} \le \sum_{l=0}^{\infty} P\{\nu_0 = 0\}$$

and $q \equiv P\{\nu_0 = 0\} > 0$. This allows us to define a new probability measure Q on $\mathfrak{B}$ by

$$Q(A) = \frac{1}{q} P\{A, \nu_0 = 0\}, \qquad A \in \mathfrak{B}.$$

We denote by L_j the "excursion" between ν_j and ν_{j+1}, i.e. L_j denotes the finite sequence $L_j = \{X_{\nu_j+1}, X_{\nu_j+2}, \cdots, X_{\nu_{j+1}}\}$. The L_j take their values in the space S of finite sequences of real numbers. The main point of the proof is that $L_0, L_1, \cdots$ is a stationary sequence under the Q measure, and that

$$\int (\nu_1 - \nu_0) dQ = 1/q < \infty. \tag{2.2}$$

To prove stationarity, note that for measurable $B_i \subset S$

$$\begin{aligned} Q\{L_1 \in B_0, \cdots, L_m \in B_{m-1}\} &= \frac{1}{q} P\{\nu_0 = 0, L_1 \in B_0, \cdots, L_m \in B_{m-1}\} \\ &= \frac{1}{q} \sum_{l=1}^{\infty} P\{\nu_0 = 0, \nu_1 = l, L_1 = B_0, \cdots, L_m \in B_{m-1}\}. \end{aligned} \tag{2.3}$$

Now, on $\{\nu_1 = l\}$ the condition $L_1 \in B_0, \cdots, L_m \in B_{m-1}$ is a condition only on $X_{l+1}, X_{l+2}, \cdots$ which can be written as $\{X_r\}_{r \ge l+1} \in C$ for some $C \subset \mathbf{R} \times \mathbf{R} \times \cdots$ which does not depend on l. In addition, it is not hard to see from (2.1) that on $\{\nu_0 = 0\}$ ν_1 is the last index at which $\min_{n\ge 1} S_n$ is achieved. This fact plus the obvious relation $S_j = S_l - (S_l - S_j)$, $1 \le j < l$, quickly yields

$$\begin{aligned} \{\nu_0 = 0, \nu_1 = l\} &= \Big\{\min_{n>0} S_n > 0, \min_{n>l} S_n - S_l > 0, \\ &\qquad \text{but for } 1 \le j < l \min_{n>j} S_n - S_j \le 0\Big\} \\ &= \Big\{\min_{n>l} S_n - S_l > 0, S_l > 0, \text{ but for } 1 \le j < l, S_l - S_j \le 0\Big\}. \end{aligned} \tag{2.4}$$

When the indices of all X_i's are reduced by l, the event in (2.4) goes over into

$$\left\{\min_{n>0} S_n > 0,\ \sum_{1-l\le i\le 0} X_i > 0,\ \text{but for } 1 \le j < l,\ \sum_{j+1-l\le i\le 0} X_i \le 0\right\}$$

$$= \{\nu_0 = 0 \text{ and } \nu^* = l\},$$

where

$$\nu^* = \min\left\{m \ge 1 : \sum_{-m+1\le i\le \nu_0} X_i > 0\right\}.$$

Since shifting indices by l leaves the P measure unchanged, we obtain from these observations and (2.3) that

$$\begin{aligned} Q\{L_1 \in B_0, \cdots, L_m \in B_{m-1}\} &= \frac{1}{q}\sum_{l=1}^{\infty} P\{\nu_0 = 0,\ \nu^* = l,\ \{X_r\}_{r\ge 1} \in C\} \\ &= \frac{1}{q} P\{\nu_0 = 0,\ \nu^* < \infty,\ \{X_r\}_{r\ge 1} \in C\}. \end{aligned} \tag{2.5}$$

But $\Sigma_{-m\le i\le 0} X_i$ has the same distribution as S_{m+1} and hence tends to ∞ in probability. Consequently $\nu^* < \infty$ a.e. $[P]$. Also, on $\{\nu_0 = 0\}$, $\{X_r\}_{r\ge 1} \in C$ is the same as $L_0 \in B_0, \cdots, L_{m-1} \in B_{m-1}$, so that by (2.5)

$$Q\{L_1 \in B_0, \cdots, L_m \in B_{m-1}\} = \frac{1}{q} P\{\nu_0 = 0,\ L_0 \in B_0, \cdots, L_{m-1} \in B_{m-1}\}$$

$$= Q\{L_0 \in B_0, \cdots, L_{m-1} \in B_{m-1}\}.$$

This demonstrates the stationarity of $L_0, L_1, \cdots$. Quite similar arguments prove (2.2). Indeed

$$\int(\nu_1 - \nu_0)\,dQ = \sum_{l=1}^{\infty} lQ\{\nu_1 - \nu_0 = l\} = \frac{1}{q}\sum_{l=1}^{\infty} lP\{\nu_0 = 0, \nu_1 = l\}$$

$$= \frac{1}{q}\sum_{l=1}^{\infty} lP\left\{\min_{n>l} S_n - S_l > 0,\ S_l > 0, \text{ but for } 1 \le j < l,\ S_l - S_j \le 0\right\}$$

$$= \frac{1}{q}\sum_{l=1}^{\infty}\sum_{k=1}^{l} P\left\{\min_{n>l-k} S_n - S_{l-k} > 0,\ \sum_{1-k\le i\le l-k} X_i > 0,\right.$$

$$\left.\text{but for } 1 \le j < l,\ \sum_{j+1-k\le i\le l-k} X_i \le 0\right\}$$

$$= \frac{1}{q}\sum_{l=1}^{\infty}\sum_{k=1}^{l} P\{\nu_0 = l-k,\ \nu^* = k\} = \frac{1}{q} P\{\nu_0 < \infty,\ \nu^* < \infty\} = \frac{1}{q}.$$

We can now apply the ergodic theorem to the sequence $L_0, L_1, \cdots$ on the probability space $(\Omega, \mathfrak{B}, Q)$. Since $\nu_{j+1} - \nu_j = f(L_j)$ for a suitable function $f: S \to \{1, 2, \cdots\}$ we obtain from (2.2) that

$$\text{(2.6)} \quad \alpha \equiv \lim_{k\to\infty} \frac{\nu_k}{k} = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j) \text{ exists and is finite a.e. } [Q]$$

(see [1, Theorem 6.21]). Similarly, we can write $S_{\nu_{j+1}} - S_{\nu_j} = g(L_j)$, and since, by definition of ν_j,

$$S_{\nu_{j+1}} - S_{\nu_j} = \sum_{\nu_j < i \le \nu_{j+1}} X_i > 0$$

we have $g > 0$. Thus, by [1, §6.6],

$$\text{(2.7)} \quad \beta \equiv \lim_{k\to\infty} \frac{S_{\nu_k}}{k} = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} g(L_j) = E^Q\{g(L_1)|\mathfrak{J}\} > 0 \text{ a.e. } [Q],$$

where $\mathfrak{J}$ is the σ-field of invariant sets, and $E^Q\{\ |\mathfrak{J}\}$ is the conditional expectation w.r.t. $\mathfrak{J} \subset \mathfrak{B}$ on the measure space $(\Omega, \mathfrak{B}, Q)$. (2.6), (2.7) plus $S_n \ge S_{\nu_k}$ for all $\nu_k \le n < \nu_{k+1}$ now imply

$$\text{(2.8)} \quad \liminf_{n\to\infty} \frac{S_n}{n} \ge \lim_{k\to\infty} \frac{S_{\nu_k}}{\nu_{k+1}} = \lim_{k\to\infty} \frac{S_{\nu_k}}{k} \frac{k}{\nu_{k+1}} = \frac{\beta}{\alpha} > 0 \text{ a.e. } [Q].$$

To complete the proof we merely have to show that Q may be replaced by P in (2.8). That this is indeed permissible follows from

$$\begin{aligned} P\left\{\liminf_{n\to\infty} \frac{S_n}{n} \le 0\right\} &= \sum_{l=0}^{\infty} P\left\{\nu_0 = l, \liminf \frac{S_n - S_l}{n} \le 0\right\} \\ &\le \sum_{l=0}^{\infty} P\left\{\min_{n>l} S_n - S_l > 0, \liminf \frac{S_n - S_l}{n} \le 0\right\} \\ &= \sum_{l=0}^{\infty} P\left\{\min_{n>0} S_n > 0, \liminf \frac{S_n}{n} \le 0\right\} \\ &= \sum_{l=0}^{\infty} P\left\{\nu_0 = 0, \liminf \frac{S_n}{n} \le 0\right\} \\ &= \sum_{l=0}^{\infty} qQ\left\{\liminf \frac{S_n}{n} \le 0\right\} = 0. \end{aligned}$$

3. **Comments:** (i) The theorem shows that $P\{S_n \to \infty\} = 0$ whenever

$$P\{\lim n^{-1}S_n = 0\} = 1, \tag{3.1}$$

or $P\{\liminf_{n\to\infty} S_n < \infty\} = 1$ whenever (3.1) holds. Without further conditions we cannot sharpen this to

$$P\left\{\liminf_{n\to\infty} S_n = -\infty\right\} = 1. \tag{3.2}$$

For example, (3.1) but not (3.2) holds when

$$P\{X_i = (-1)^i,\ i = 1, 2, \cdots\} = P\{X_i = (-1)^{i-1},\ i = 1, 2, \cdots\} = ½.$$

In this example P is concentrated on two sequences each with $|S_n|$ bounded. The following example shows that we cannot even obtain $\liminf S_n = -\infty$ a.e. on the set $\{\limsup S_n = +\infty\}$. Let $Y_0, Y_1, \cdots$ be a stationary sequence of positive functions, for which

$$\int Y_1 dP < \infty \quad \text{and} \quad P\{\limsup Y_n = \infty\} = 1$$

(e.g. the Y_i could be independent, identically distributed random variables). If we put $X_i = Y_i - Y_{i-1}$, $i \geq 1$, then also $\{X_i\}_{i\geq 1}$ is stationary and $S_n = Y_n - Y_0 \geq -Y_0$. Thus the probability in (3.2) equals zero while (3.1) holds and $P\{\limsup S_n = \infty\} = 1$.

(ii) It would be of interest to replace $\liminf n^{-1}S_n$ by $\lim n^{-1}S_n$ in the theorem. This is permissible when

$$\limsup_{k\to\infty} \frac{1}{k} \max_{\nu_k \leq n < \nu_{k+1}} S_n - S_{\nu_k} = \limsup_{k\to\infty} \frac{1}{k} \max_{j < \nu_{k+1} - \nu_k} \sum_{i=1}^{j} X_{\nu_k + i} = 0. \tag{3.3}$$

A recent result of Tanny [2] shows that the random variable in the left-hand side of (3.3) takes only the values 0 and ∞ a.e. This, together with the proof of §2 shows that one has $\lim n^{-1}S_n > 0$ or

$$0 < \liminf n^{-1}S_n \leq \limsup n^{-1}S_n = +\infty$$

a.e. on $\{S_n \to \infty\}$.

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$$\limsup \frac{S_n}{n} = +\infty \quad \text{or} \quad \liminf \frac{S_n}{n} = -\infty \quad \text{a.e.}$$

on the set where $\lim n^{-1}S_n$ does not exist. Equivalently

$$P\{-\infty < \liminf \frac{S_n}{n} < \limsup \frac{S_n}{n} < +\infty\} = 0.$$

(Indeed, if $\liminf S_n/n \geq -k > -\infty$, then

$$\liminf n^{-1} \sum_{i=1}^{n} (X_i + 2k) > 0$$

and hence by comment (ii) either

$$\lim n^{-1} \sum_{i=1}^{n} (X_i + 2k)$$

exists or $\limsup n^{-1} \sum_{i=1}^{n} (X_i + 2k) = +\infty$.)

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