THE ASYMPTOTIC BEHAVIOUR OF THE REDUCED MINIMUM MODULUS OF A FREDHOLM OPERATOR

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ABSTRACT. Let \( \gamma(S) \) denote the reduced minimum modulus of a linear operator \( S \) acting in a complex Banach space \( X \), and let \( I \) denote the identity on \( X \). In this paper it is shown that for a (not necessarily bounded) Fredholm operator \( T \) acting in \( X \), the limit \( \lim_{n \to \infty} \gamma(T^n)^{1/n} \) exists and is equal to the supremum of all positive numbers \( \delta \) such that the dimension of the null space and the codimension of the range of \( T - \lambda I \) are constant on \( 0 < |\lambda| < \delta \).

Introduction. Throughout this paper \( T \) will be a linear operator with domain \( D(T) \) and range \( R(T) \) in the complex Banach space \( X \). By definition the reduced minimum modulus \( \gamma(T) \) of \( T \) is the supremum of all real numbers \( \gamma \) such that

\[
\|Tx\| \geq \gamma d(x, N(T)), \quad x \in D(T)
\]

(cf. [6, p. 231] and [3, Definition IV.1.3]). Here \( d(x, N(T)) \) denotes the distance of \( x \) to the null space \( N(T) \) of \( T \). Observe that we do not require \( N(T) \) to be closed in \( X \). In [3] \( \gamma(T) \) is called the minimum modulus of \( T \), but in the present paper this term is reserved for the object studied in [2].

Let \( n(T) \) denote the dimension of \( N(T) \) and \( d(T) \) the codimension of \( R(T) \) in \( X \). We call \( T \) a Fredholm operator if \( T \) is a closed linear operator with \( n(T) \) and \( d(T) \) both finite. If \( T \) is a closed linear operator with closed range such that at least one of the numbers \( n(T) \) and \( d(T) \) is finite, then \( T \) is said to be a semi-Fredholm operator. Since the range of a Fredholm operator is closed (cf. [5, Lemma 332]), any Fredholm operator is semi-Fredholm.

In §1 we show that for a semi-Fredholm operator \( T \)

\[
\lim_{n \to \infty} \gamma(T^n)^{1/n}
\]

exists. Further we compute the limit (1) for a few examples.

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Suppose that $T$ is a Fredholm operator. From stability theory (cf. §IV.5 in [6]) we know that there exists a positive number $\delta$ such that $n(T - \lambda I)$ and $d(T - \lambda I)$ are constant on $0 < |\lambda| < \delta$. In §2 we shall show that the supremum of all $\delta$ with this property is equal to the limit (1). This result is based on a further elaboration of the stability theorems involved and on the existence of holomorphic relative inverses for certain holomorphic operator valued functions (see [1]). As a corollary we obtain a generalization of [2, Theorem 3.5] to the Fredholm set of an operator.

1. The existence of $\lim \gamma(T^n)^{1/n}$.

**Lemma 1.** Suppose that $N(T^k) \subseteq R(T)$ for $k = 1, 2, \ldots$. Then

\[(2) \quad \gamma(T^{n+m}) \geq \gamma(T^n)\gamma(T^m) \quad (n, m = 1, 2, \ldots).\]

**Proof.** Take $x$ in $D(T^{n+m})$. For each $u$ in $N(T^{n+m})$, we have

\[(3) \quad d(x, N(T^{n+m})) = d(x - u, N(T^{n+m})) \leq d(x - u, N(T^m)).\]

By induction it follows from our hypotheses that $T^m[N(T^{n+m})] = N(T^n)$. Using this together with (3), we see that

\[d(T^m x, N(T^n)) = \inf\{\|T^m(x - u)\|: u \in N(T^{n+m})\} \geq \gamma(T^m) \inf\{d(x - u, N(T^m)): u \in N(T^{n+m})\} \geq \gamma(T^m)d(x, N(T^{n+m})),\]

and hence

\[\|T^{n+m}x\| \geq \gamma(T^n)d(T^m x, N(T^n)) \geq \gamma(T^n)\gamma(T^m)d(x, N(T^{n+m})).\]

This proves the lemma.

The inequality (2) implies that $\lim \gamma(T^n)^{1/n}$ exists (see [9, Problem 98]), but not necessarily as a finite number (see Example (1) below). In fact

\[\rho(T) = \lim_{n \to \infty} \gamma(T^n)^{1/n} = \sup_{n \geq 1} \gamma(T^n)^{1/n}.\]

In particular, this limit exists if $T$ is injective, and in that case $\rho(T) = \lim_{n \to \infty} \mu(T^n)^{1/n}$, where $\mu(T)$ is the minimum modulus as defined in [2].

**Theorem 2.** If $T$ is a semi-Fredholm operator, then

\[\rho(T) = \lim_{n \to \infty} \gamma(T^n)^{1/n}\]

exists.

**Proof.** From our hypotheses it follows that we may apply Theorem 4 in
[5] to show that $X$ decomposes into two $T$-invariant closed subspaces $X_0$ and $X_1$ which have the following properties. The space $X_1$ is finite dimensional and $X_1 \subseteq \mathcal{N}(T^k)$ for some $k$. If $T_0$ is the restriction of $T$ to $X_0$ considered as an operator from $X_0$ into itself, then $\mathcal{N}(T_0^n) \subseteq \mathcal{R}(T_0)$ for $n = 1, 2, \cdots$.

This last fact together with Lemma 1 implies that $\rho(T_0) = \lim \gamma(T_0^n)^{1/n}$ exists. We shall show that $\rho(T) = \rho(T_0)$.

Choose $k$ such that $X_1 \subseteq \mathcal{N}(T^k)$, and let $P$ be the bounded projection of $X$ along $X_1$ onto $X_0$. Take $n \geq k$. Since $X_1 \subseteq \mathcal{N}(T^k) \subseteq \mathcal{N}(T^n) \subseteq \mathcal{D}(T^n)$, it follows that $x \in \mathcal{D}(T^n)$ if and only if $Px \in \mathcal{D}(T^n)$, and in that case

$$T^n x = T^n_0 Px.$$  

Further, we have for each $x$ in $X$

$$d(x, \mathcal{N}(T^n)) \leq d(Px, \mathcal{N}(T^n)) \leq \|P\|d(x, \mathcal{N}(T^n)).$$  

Together with (4), this implies that

$$\gamma(T_0^n) \leq \gamma(T^n) \leq \|P\|\gamma(T^n),$$

and hence

$$\lim_{n \to \infty} \gamma(T_0^n)^{1/n} = \lim_{n \to \infty} \gamma(T^n)^{1/n}.$$  

This proves the theorem.

**Examples.** (1) Let $X = L_p([a, b])$ with $-\infty < a < b < +\infty$ and $1 \leq p \leq \infty$. Let $T$ be the linear operator with domain the set of all absolutely continuous functions $f$ on $[a, b]$ such that $f$ and its derivative $f'$ belong to $L_p([a, b])$, and suppose that $Tf = f'$, $f \in \mathcal{D}(T)$. Then $T$ is a surjective linear operator (cf. [3, Theorem VI.3.1]). Thus Lemma 1 implies that $\rho(T)$ exists. By induction one can show that $\mathcal{D}(T^n)$ is the set of all $f \in L_p([a, b])$ such that the $(n-1)$th derivative $f^{(n-1)}$ is absolutely continuous on $[a, b]$ and $f^{(n)} \in L_p([a, b])$. Further, $T^n f = f^{(n)}$ for $f \in \mathcal{D}(T^n)$. Hence we can use Example IV.1.4 in [3] to show that $\gamma(T^n) \geq n!/(b-a)^n$, $n = 1, 2, \cdots$. Thus $\rho(T) = \infty$.

(2) The inequality (2) does not hold in general; not even if $T$ satisfies the conditions of Theorem 2. To see this take $X = C_k$ with $k > 1$ and the Euclidean norm, and define $T$: $X \to X$ by $Tx = (x_1, x_2, \cdots, x_k)$, where $x = (x_1, x_2, \cdots, x_k)$. Then $\|Tx\| = \sqrt{k}|x_1|$ and $d(x, \mathcal{N}(T)) = |x_1|$ for all $x$. Thus $\gamma(T) = \sqrt{k}$. Since $T^2 = T$, we have $\gamma(T^n) = \sqrt{k}$ for $n = 1, 2, \cdots$. As $k > 1$, this implies that (2) does not hold in this case.

There are other cases in which $\lim \gamma(T^n)^{1/n}$ exists than those covered by Lemma 1 and Theorem 2. For instance, if $T$ is everywhere defined,
bounded and nilpotent, then $\gamma(T^n) = +\infty$ for $n$ sufficiently large, and it follows that $\lim\gamma(T^n)^{1/n} = +\infty$. If $X$ is a Hilbert space and $T$ a selfadjoint linear operator, then $\gamma(T^n) = \gamma(T)^n$ for $n = 1, 2, \ldots$. Hence in that case $\rho(T) = \gamma(T)$.

2. Application to Fredholm operators. Define $k(T)$ to be the dimension of the quotient space

$$N(T)/\left[\bigcap_{n=1}^{\infty} R(T^n)\right] \cap N(T)$$

(see [4, Definition 2.1]). The condition $k(T) = 0$ is equivalent to the requirement that $N(T^n) \subset R(T)$ for $n = 1, 2, \ldots$. If $T$ is a Fredholm operator, then $k(T) < +\infty$. We need the following special case of Theorem 5.1 in [4].

Theorem 3. If $T$ is a Fredholm operator, then there exists a positive number $\delta$ such that for $0 < |\lambda| < \delta$

(a) $T - \lambda I$ is a Fredholm operator with $k(T - \lambda I) = 0$,
(b) $n(T - \lambda I) = n(T) - k(T)$,
(c) $d(T - \lambda I) = d(T) - k(T)$.

In the particular case that $k(T) = 0$, the constant $\delta$ may be taken to be $\gamma(T)$.

Let $T$ be a Fredholm operator. Define $\delta(T)$ to be the supremum of all $\delta > 0$ such that $T - \lambda I$ is a Fredholm operator and $k(T - \lambda I) = 0$ for $0 < |\lambda| < \delta$. Since $T$ is Fredholm, it follows from Theorem 3 that $\delta(T)$ is the greatest positive number $\delta$ (possibly $\infty$) such that $n(T - \lambda I)$ and $d(T - \lambda I)$ are constant on $0 < |\lambda| < \delta$. The next theorem shows that

$$\delta(T) = \rho(T) = \lim_{n \to \infty} \gamma(T^n)^{1/n}.$$ 

In the proof of this theorem we need the following lemma.

Lemma 4. Let $S \colon X \to X$ be a bounded linear operator, and suppose that $TST = T$. Then $\gamma(T) \geq \|S\|^{-1}$.

Proof. Take $x \in D(T)$. Since $S$ is everywhere defined, $x \in D(ST)$. Further, it follows from $TST = T$ that $Tx = T(STx)$, and thus $x - STx \in N(T)$. But then

$$d(x, N(T)) = d(STx, N(T)) \leq \|STx\| \leq \|S\| \cdot \|Tx\|.$$ 

This proves the lemma.
Theorem 5. If $T$ is a Fredholm operator, then

$$\lim_{n \to \infty} \gamma(T^n)^{1/n}$$

exists and is equal to the supremum of all $\delta > 0$ such that $n(T - \lambda I)$ and $d(T - \lambda I)$ are constant on $0 < |\lambda| < \delta$.

Proof. From Theorem 2 we know that $\rho(T) = \lim \gamma(T^n)^{1/n}$ exists. Thus we have to prove the first equality in (6). In order to do this, we first consider the case $k(T) = 0$.

Suppose $k(T) = 0$. Take $0 < |\lambda| < \rho(T)$. Then there exists a positive integer $l$ such that $0 < |\lambda^l| < \gamma(T^l)$. Since $T$ is a Fredholm operator, the same is true for $T^l$. Further, $k(T) = 0$ implies $k(T^l) = 0$. Hence we can apply Theorem 3 to show that $T^l - \lambda^lI$ is a Fredholm operator and $k(T^l - \lambda^lI) = 0$. So we have

$$n(T - \lambda I) \subseteq n(T^l - \lambda^lI) \subseteq R((T^l - \lambda^lI)^n) \subseteq R((T - \lambda I)^n)$$

for $n = 1, 2, \cdots$. But then it follows that $k(T - \lambda I) = 0$ and $n(T - \lambda I)$ and $d(T - \lambda I)$ are finite. Since $T$ is closed, the operator $T - \lambda I$ is closed, and thus $T - \lambda I$ is a Fredholm operator with $k(T - \lambda I) = 0$. This shows that $\rho(T) \leq \delta(T)$.

To prove the reverse inequality for $k(T) = 0$, we have to apply a result on bounded operators. Let $G$ denote the domain $D(T)$ endowed with the graph norm. Then $G$ is a complex Banach space, and, if $\kappa$ is the canonical imbedding of $G$ into $X$, then $\kappa$ and $T\kappa$ are bounded linear operators from $G$ into $X$. Observe that

$$n(T - \lambda I) = n(T\kappa - \lambda\kappa), \quad d(T - \lambda I) = d(T\kappa - \lambda\kappa).$$

In particular, $T - \lambda I$ is Fredholm if and only if $T\kappa - \lambda\kappa$ is Fredholm.

Let $\Delta = \{\lambda \in \mathbb{C} : |\lambda| < \delta(T)\}$. From the definition of $\delta(T)$ and the results of the previous paragraph it follows that $\lambda \mapsto T\kappa - \lambda\kappa$ is a holomorphic Fredholm operator valued function on $\Delta$. Since $k(T) = 0$, we have

$$n(T\kappa - \lambda\kappa) = n(T - \lambda I) = n(T), \quad \lambda \in \Delta.$$ 

Hence, by [1, Theorem 5.2], there exists a holomorphic function $B$ on $\Delta$ such that $B(\lambda)$ is a bounded linear operator from $X$ into $G$ and

$$(T\kappa - \lambda\kappa)B(\lambda)(T\kappa - \lambda\kappa) = T\kappa - \lambda\kappa$$

for all $\lambda \in \Delta$. Define on $\Delta$ the holomorphic function $Q$ by

$$Q(\lambda) = I_G - B(\lambda)(T\kappa - \lambda\kappa),$$
where \( I_G \) denotes the identity operator on \( G \). Observe that for each \( \lambda \) in \( \Delta \), the operator \( Q(\lambda) \) is a bounded linear operator on \( G \). From (7) it follows that

\[
(T \kappa - \lambda \kappa)Q(\lambda) = 0, \quad \lambda \in \Delta.
\]

The holomorphic functions \( B \) and \( Q \) have power series expansions on \( \Delta \).

Let

\[
B(\lambda) = \sum_{n=0}^{\infty} B_n \lambda^n, \quad Q(\lambda) = \sum_{n=0}^{\infty} Q_n \lambda^n.
\]

For each \( n \), the operator \( B_n \) is a bounded linear operator from \( X \) into \( G \) and \( Q_n \) is a bounded linear operator on \( G \). Since the series in (10) converge for all \( \lambda \) in \( \Delta \), we have

\[
\limsup \| B_n \|^{1/n} \leq \delta(T).
\]

From the formulas (8) and (9) it follows that there exist several relations between the coefficients of the power series in (10) and the operator \( T \). We shall use these relations to show that the left-hand side of (11) is less than \( \rho(T) \).

From (9) we see that

\[
TkQ_n = 0, \quad T \kappa Q_n = \kappa Q_{n-1}, \quad n = 1, 2, \ldots
\]

By induction this implies that

\[
R(\kappa Q_n) \subset D(T^{n+1}), \quad T^{n+1} \kappa Q_n = 0
\]

for \( n = 0, 1, 2, \ldots \). From formula (8), we see that

\[
Q_0 = I_G - B_0 T \kappa, \quad Q_n = -B_n T \kappa + B_{n-1} \kappa
\]

for \( n = 1, 2, \ldots \). Take a fixed nonnegative integer \( k \), and let \( x \in D(T^{k+1}) \). Then \( T^n x \in D(T) \) for \( n = 0, 1, \ldots, k \). Now \( G \) and \( D(T) \) considered as sets are equal. Thus \( T^n x \in G \) and \( \kappa(T^n x) = T^n x \) for \( n = 0, 1, \ldots, k \). Hence, using (14),

\[
Q_0 x = x - B_0 T x
\]

and

\[
Q_n T^n x = -B_n T^{n+1} x + B_{n-1} T^n x, \quad n = 1, \ldots, k.
\]

We shall show that this implies that

\[
T^{n+1} \kappa Q_n T^n x = T^{n+1} x, \quad n = 0, 1, \ldots, k.
\]
For \( n = 0 \), this follows from (15) and the first part of (12). Proceeding by induction, suppose that \( T^n \kappa B_{n-1} T^n x = T^n x \) for some \( n \) with \( 0 \leq n - 1 < k \). Since \( x \in D(T^{-1}) \subset D(T^{n+1}) \), we have \( T^n x \in D(T) \). Thus \( T^n (\kappa B_{n-1} T^n x) \in D(T) \), and it follows that \( \kappa B_{n-1} T^n x \in D(T^{n+1}) \). Further,

\[
T^{n+1} \kappa B_{n-1} T^n x = T^{n+1} x.
\]

Also \( \kappa Q_n T^n x \in D(T^{n+1}) \) by (13). Using this in (16), we see that \( \kappa B_n T^{n+1} x \in D(T^{n+1}) \). But then it follows from the second part of (13), (16) and (18) that (17) holds. In particular

\[
T^{k+1} \kappa B_k T^{k+1} = T^{k+1}.
\]

So we can apply Lemma 4 to show that

\[
\gamma(T^{k+1}) \geq \| \kappa B_k \|^{-1} \geq \| \kappa \|^{-1} \| B_k \|^{-1}.
\]

Since \( \| \kappa \| \leq 1 \), this implies that

\[
\gamma(T^{k+1}) \geq \| B_k \|^{-1}, \quad k = 0, 1, 2, \ldots.
\]

Using this in (11), it follows that \( \rho(T) \geq \delta(T) \). Thus \( \rho(T) = \delta(T) \) if \( k(T) = 0 \).

If \( k(T) \neq 0 \), we apply the reduction process which was used to prove Theorem 2. Let \( X_0, X_1 \) and \( T_0 \) be as in the proof of that theorem, and let \( I_0 \) denote the identity operator on \( X_0 \). Since \( X_1 \) is finite dimensional, the operator \( \lambda I_0 - T_0 \) is a Fredholm operator if and only if \( \lambda I - T \) is Fredholm. Hence our hypotheses imply that \( T_0 \) is Fredholm. Further we know that \( \mathcal{N}(T_0^n) \subset \mathcal{R}(T_0^n) \) for \( n = 1, 2, \ldots \). But this implies that \( \mathcal{N}(T_0^n) \subset \mathcal{R}(T_0^n) \) for \( n = 1, 2, \ldots \), and hence \( k(T_0) = 0 \). Therefore, by the result proved above, \( \rho(T_0) = \delta(T_0) \). We already know that \( \rho(T) = \rho(T_0) \) (see formula (5)). Hence in order to prove the theorem, it remains to show that \( \delta(T) = \delta(T_0) \).

Let \( T_1 \) be the restriction of \( T \) to \( X_1 \) considered as an operator from \( X_1 \) into \( X_1 \), and let \( I_1 \) denote the identity operator on \( X_1 \). Since \( X_1 \subset \mathcal{N}(T^k) \) for some \( k \), the operator \( T_1 \) is defined on all of \( X_1 \) and \( T_1 \) is bounded and nilpotent. This implies that \( \lambda I_1 - T_1 \) is bijective for \( \lambda \neq 0 \). Hence, for \( \lambda \neq 0 \),

\[
\mathcal{N}(\lambda I - T) = \mathcal{N}(\lambda I_0 - T_0),
\]

\[
\mathcal{R}((\lambda I - T)^n) = \mathcal{R}((\lambda I_0 - T_0)^n) \oplus X_1, \quad n = 1, 2, \ldots.
\]

But then it follows that \( k(\lambda I - T) = 0 \) if and only if \( k(\lambda I_0 - T_0) = 0 \) for \( \lambda \neq 0 \).
We have already observed that $T - \lambda I$ is Fredholm if and only if $T_0 - \lambda I_0$ is Fredholm. Thus $\delta(T) = \delta(T_0)$, and the proof is complete.

By definition the Fredholm set $\Phi(T)$ of $T$ is the set of all complex numbers $\lambda$ such that $T - \lambda I$ is a Fredholm operator. Let

$$\Delta(T) = \{\lambda \in \Phi(T) : k(T - \lambda I) = 0\}.$$ 

By Theorem 3, the sets $\Phi(T)$ and $\Delta(T)$ are both open in $\mathbb{C}$. If $V$ is a subset of $\mathbb{C}$, we let $d(\lambda, V)$ denote the distance in $\mathbb{C}$ of the point $\lambda$ to $V$.

With this notation we have the following corollary to Theorem 5.

**Corollary 6.** If $\lambda \in \Phi(T)$, then

$$\lim_{n \to \infty} \gamma((T - \lambda I)^n)^{1/n} = d(\lambda, \mathbb{C} \setminus \Delta(T) \cup \{\lambda\}).$$

Let $T$ be closed, and suppose that $\lambda$ does not belong to the spectrum $\sigma(T)$ of $T$ (i.e., $T - \lambda I$ is bijective). Then one can use Theorem 3 to show that

$$d(\lambda, \mathbb{C} \setminus \Delta(T) \cup \{\lambda\}) = d(\lambda, \sigma(T)).$$

This implies that Theorem 3.5 of [2] is a special case of the above corollary (cf. the remark preceding Theorem 2).

Let $K(X)$ denote the set of compact linear operators on $X$. Define

$$\gamma_c(T) = \sup_{K \in K(X)} \inf \{ \|Tx\|/\|x - Kx\| : x \in D(T), Kx \neq x \}.$$ 

Roughly speaking $\gamma_c(T)$ is the reduced minimum modulus of $T$ corresponding to the $m$-seminorm introduced by A. Lebow and M. Schechter in [7]. E.-O. Liebetrau [8] has used Theorem 5 of the present paper (for the case that $k(T) = 0$) to show that for $\lambda \in \Phi(T)$

$$\lim_{n \to \infty} \gamma_c((T - \lambda I)^n)^{1/n} = d(\lambda, \mathbb{C} \setminus \Phi(T)).$$

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CHARACTERIZATIONS OF THE COMPLEX PROJECTIVE PLANE BY CURVATURE

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ABSTRACT. We give short proofs to show that under various positivity assumptions on the curvature of a Kähler surface $X$, it is biholomorphically equivalent to $\mathbb{P}^2(\mathbb{C})$. In particular, the case of $\delta$-holomorphic pinching $>\frac{1}{2}$ (Theorem 1) is best possible and, we believe, new.

Let $X$ be a compact Kähler manifold of complex dimension 2. Given a real two-dimensional subspace $p$ of the real tangent space of $X$ at some point, we denote by $K(p)$ the Riemannian curvature of $p$. $X$ is called $\delta$-holomorphically pinched if there is some constant $A > 0$ such that $\delta A \leq K(p) \leq A$ for all holomorphic planes $p$, i.e. for all planes $p$ invariant under the complex structure endomorphism $J$. Given two holomorphic planes...