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COINCIDENCE POINT RESULTS FOR SPACES WITH FREE Z_p- ACTIONS

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ABSTRACT. Let X support a free cyclic group action of prime order. We consider the question of determining when any map $f: X \to Y$ must identify two points of an orbit, and that of finding the minimum possible dimension of the union of such orbits when they exist.

1. **Introduction.** Let X be a path-connected Hausdorff space which supports a free π_p-action, where π_p is the cyclic group of prime order p. Let Y be any space and consider the question of whether there exists for any map $f: X \to Y$ a point $x \in X$ such that $f(x) = f(\sigma^i x)$ for some $i \neq 0$. The classical Borsuk-Ulam Theorem gives an affirmative answer for the case in which $p = 2$, $X = S^n$, and $Y = R^n$. Various generalizations have appeared in [5], [9], [4], and [3].

We are particularly interested in the line of inquiry begun in [4], and present here some extensions of the above-mentioned results to the case in which p is an arbitrary prime and Y is a path-connected complex or manifold of dimension at least 2. (The assumptions on Y can be weakened; they are present in order to facilitate explicit calculations.) Let $f: X \to Y$ and define

$$A(X, f) = \{x \in X \mid f(\sigma^i x) = f(\sigma^j(x)) \text{ for some } i \neq j,\ 0 \leq i, j \leq p - 1\}.$$

We will define a number N depending on Y and p and prove the following two theorems. All cohomology is taken with Z_p coefficients.

We would like to thank the referee for pointing out the present (and much better) version of Lemma 2 and for the remarks which are contained in §6.

Theorem 1. *If X is a path-connected Hausdorff space and $H^i(X) = 0$ for $0 < i \leq N$, then $A(X, f) \neq \emptyset$.*

Theorem 2. *If X is a closed m-manifold with $H^i(X) = 0$ for $0 \leq i \leq N$, then* $\dim A(X, f) \geq m - N - 1$.

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Remark. Here $\dim A$ means the covering dimension of A. By the argument which appears in [9], to prove $\dim A \geq k$ it is sufficient to prove that $\overline{H}^k(A) \neq 0$, where $\overline{H}$ denotes Alexander-Spanier cohomology.

We now define N. Consider the subspace $F(Y, p) = \{(y_1, \cdots, y_p) \mid y_i \neq y_j \text{ for } i \neq j\}$ of Y^p. There is a free π_p-action on $F(Y, p)$ given by cyclic permutation of coordinates. These spaces have been studied in [2]. If $E\pi_p$ is a contractible free π_p-space, there is an equivariant map ϕ inducing the following covering space maps:

$$\begin{array}{ccc} F(Y, p) & \xrightarrow{\phi} & E\pi_p \\ \downarrow & & \downarrow \\ F(Y, p)/\pi_p & \xrightarrow{\hat{\phi}} & B\pi_p \end{array}$$

The classifying space $B\pi_p$ is known to have nonzero mod p cohomology in all dimensions. We define $N = N(Y, p)$ to be the largest integer such that $\hat{\phi}^* H^N(B\pi_p) \neq 0$.

Some estimates on N for certain spaces Y are given in §3 below. In §4 are some examples which give bounds on possible improvements of the main theorems.

2. **Proofs of Theorems 1 and 2.** Define $\psi: X \to Y^p$ by

$$\psi(x) = (f(x), f(\sigma x), \cdots, f(\sigma^{p-1} x)).$$

Proof of Theorem 1. If $A(X, f) = \varnothing$, then ψ is an equivariant map of X into $F(Y, p)$. Consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\psi} & F(Y, p) & \xrightarrow{\phi} & E\pi_p \\ \downarrow & & \downarrow & & \downarrow \\ X/\pi_p & \xrightarrow{\hat{\psi}} & F(Y, p)/\pi_p & \xrightarrow{\hat{\phi}} & B\pi_p. \end{array}$$

Since $H^i(X) = 0$ for $i \leq N$, it follows from the naturality of the spectral sequence for a covering (see [1]) that $(\hat{\phi}\hat{\psi})^*$ is a monomorphism in degrees less than or equal to $N + 1$. This contradicts the fact that $\hat{\phi}^* H^{N+1}(B\pi_p) = 0$.

Proof of Theorem 2. Observe that ψ restricts to a map of $X - A(X, f)$ into $F(Y, p)$. We may assume that $X - A(X, f)$ is path-connected. By the above argument, $H^j(X - A(X, f)) \neq 0$ for some $0 < j \leq N$, and hence $H_j(X - A(X, f)) \neq 0$ for some $0 < j \leq N$. By Alexander duality $\overline{H}^{m-j}(X, A(X, f)) \neq 0$. Similarly $H^j(X) = 0$ implies $\overline{H}^{m-j}(X) = 0$ and therefore by the cohomology exact sequence

$$\bar{H}^{m-j-1}(A(X, f)) \neq 0.$$

The result follows.

3. **Estimation of** N. In this section we give some estimates of N for some spaces Y. The following lemma, whose proof is postponed until §5, is useful.

Lemma 1. *If* $H^i(F(Y, p)) = 0$ *for* $i \geq K$ *and* $H^1(F(Y, p)) = 0$, *then* $N(Y, p) < K$.

This means that an upper bound for $N(Y, p)$ can be found by finding the maximum dimension for which $H^i(F(Y, p)) \neq 0$. If Y is a manifold this is easy to do, using the Serre spectral sequence for the fibration

$$F(Y - pt, j - 1) \to F(Y, j) \to Y.$$

Some specific examples which have been calculated by this method are given in the table below. For each X appearing there we give an r such that if X is any r-connected space supporting a free π_p-action generated by σ then there is an $x \in X$ such that $f(x) = f(\sigma^i x)$ for some i.

Y	$r(\geq N(Y, p))$
R^n	$(n-1)(p-1)$
S^n	$(n-1)(p-1)+1$
$S^n \times R^m$	$pm + (n-1)(p-1)$
$S^n \times S^m$, $m \geq n$	$pm + (n-1)(p-1)+1$

Further, observe that $N(Y', p) \leq N(Y, p)$ if Y' embeds in Y. Consequently

$$N(Y, p) \leq [\mathrm{ED}(Y) - 1](p-1),$$

where ED is the embedding dimension of Y. Hence one gets estimates of $N(Y, p)$ for spaces such as $\mathbf{RP}^n$ and in some cases improved estimates for $S^n \times R^m$ and $S^n \times S^m$.

4. **Examples.** Here we give some examples to show that in some senses our results are the best possible for arbitrary X.

Example 1. Let $X = S^3 \times S^3$ with π_2-action $\sigma(x, y) = (-x, y)$. Define f: $S^3 \times S^3 \to S^3$ to be quaternionic multiplication. Then $f(x, y) \neq f(-x, y)$ for all $(x, y) \in S^3 \times S^3$, so $A(X, f) = \varnothing$ although $H^i(X) = 0$ for $0 < i \leq N(S^3, 2) - 1 = 2$. Consequently Theorem 1 is the best possible in the sense that N cannot be replaced by $N - 1$.

Example 2. Define f: $S^3 \to R^2$ by $f(x_1, x_2, x_3, x_4) = (x_1, x_2)$ and let

S^3 have the antipodal π_2-action. It is easy to see that $A(S^3, f) \cong S^1$ and $N(R^2, 2) = 1$. Therefore Theorem 2 is the best possible in the sense that $m - N - 1$ cannot be replaced by $m - N$; i.e., N cannot be replaced by $N - 1$ in the conclusion.

In case one puts more restrictive hypotheses on X, then the results of Theorems 1 and 2 may not be the best possible. For instance if $Y = RP^2$ and $X = S^n$ with antipodal action, then it follows from [5] that $\dim A \geq n - 2$, although Theorem 2 only gives $\dim A \geq n - 4$.

For odd primes, it may be possible to improve the theorems, but not very much. The following example, with $p = 3$, shows that N cannot be replaced by $N - 2$ in Theorems 1 and 2.

Example 3. Regard S^3 as $\{(z_1, z_2) \in \mathbf{C} \times \mathbf{C} \,|\, |z_1|^2 + |z_2|^2 = 1\}$ and let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Define $\sigma\colon S^3 \to S^3$ by

$$\sigma(z_1, z_2) = (e^{2\pi i/3} z_1, e^{2\pi i/3} z_2),$$

and $f\colon S^3 \to R^2$ by $f(z_1, z_2) = (x_1, y_1)$. Then σ generates a π_3-action on S^3 and

$$A(S^3, f) = \{(x_1, y_1, x_2, y_2) | y_1 = -\sqrt{3}x_1, y_2 = -\sqrt{3}x_2\} \cong S^1.$$

We have $H^i(S^3) = 0$ for $0 < i \leq N(R^2, 3)$, but the dimension of A is not greater than or equal to $m - (N - 2) - 1$. If we define $g\colon S^3 \to R^1$ by

$$g(x_1, y_1, x_2, y_2) = (y_1 + \sqrt{3}x_1)^2 + (y_2 + \sqrt{3}x_2)^2$$

and set

$$h(x_1, y_1, x_2, y_2) = (f(x_1, y_1, x_2, y_2), g(x_1, y_1, x_2, y_2)),$$

then $h\colon S^3 \to R^3$ and $A(S^3, h) = \emptyset$ although $H^i(S^3) = 0$ for $0 < i \leq 2 = N(R^3, 3) - 2$. Similar maps $f\colon S^{2n-1} \to R^n$ and $h\colon S^{2n-1} \to R^{n+1}$ can be constructed for larger p and n. They show that in general N cannot be replaced by $N - (n(p - 3) + 2)$.

Example 4. As in Example 3, regard S^3 as

$$\{(z_1, z_2) \in \mathbf{C} \times \mathbf{C} | \, |z_1|^2 + |z_2|^2 = 1\}.$$

Define $f\colon S^3 \to S^2$ to be the Hopf map, which takes a pair (z_1, z_2) to its equivalence class under the relation $(z_1, z_2) \sim (z_1', z_2')$ if there is a $\tau \in \mathbf{C}$ with $|\tau| = 1$ such that $z_1 = \tau z_1'$, $z_2 = \tau z_2'$. Let $z_j = x_j + iy_j$, $j = 1, 2$, and define $\lambda\colon S^3 \to S^3$ by $\lambda(z_1, z_2) = (x_1 + ix_2, y_1 + iy_2)$. Let $w = a + ib$ be any complex number with $|a|^2 + |b|^2 = 1$ and $b \neq 0$. Define $\sigma\colon S^3 \to S^3$ by

$\sigma(z_1, z_2) = (wz_1, \overline{w}z_2)$, and finally define $\rho: S^3 \to S^3$ to be $\lambda\sigma\lambda$. By evaluating the determinant of the appropriate 4×4 matrix, one can show that $\rho(z_1, z_2) \neq \tau \cdot (z_1, z_2)$ for all $(z_1, z_2) \in S^3$ and τ such that $|\tau| = 1$. For any odd prime p we let $w = e^{2\pi i/p}$, so that ρ generates a π_p-action. With this action on S^3 and this $f: S^3 \to S^2$, we have $A(S^3, f) = \emptyset$. (Of course for $p > 3$ we must apply the above calculation for powers of w as well, but since p is odd all powers have nonzero imaginary part, so the calculation goes through.) For $p = 3$ this example shows that N cannot be replaced by $N - 1$ in Theorem 1.

Example 5. If Y is a connected open 2-manifold, then by Corollary 3 of [4], whenever $\pi_1(X)$ is torsion the conclusion of Theorem 1 holds. Evidently Theorems 1 and 2 can be improved when Y is a connected open 2-manifold.

5. **Proof of Lemma 1.** Our proof uses a lemma which may be of independent interest and which describes the differentials in the spectral sequence for a covering. We will present the argument in a more general setting than that required for Lemma 1.

Recall that a finite group G is said to have p-period d if $\hat{H}^i(G; A)$ and $\hat{H}^{i+d}(G; A)$ have isomorphic p-primary components for all i and A, where $\hat{H}^i(G; A)$ is the Tate cohomology of G with coefficients in A. We assume that the periodicity homomorphism

$$\phi: \hat{H}^i(G; A) \to \hat{H}^{i+d}(G; A)$$

is given by $\phi(m) = m \cup u$ for u a fixed element in $\hat{H}^d(G; \mathbf{Z}_p)$, where "$\cup$" denotes the internal cup product and G acts trivially on $\mathbf{Z}_p$. Some examples are $\mathbf{Z}_p$, in which case $d = 2$ [8], and all subgroups of Σ_j, the symmetric group on j letters, for $j < 2p$ [11].

If G acts freely on X, there is a spectral sequence for the covering $X \to X/G$, which is a spectral sequence of algebras, converging to $H^*(X/G)$ with $E_2^{**} \approx H^*(G, H^*(X))$. Let $[x]_r$ denote a class in E_r which is represented by the class $[x]_2$ in E_2 (of course $[x]_r$ may not be defined for all $[x]_2$ in E_2). Observe that we may consider the class u in $H^d(G; \mathbf{Z}_p)$ which induces the periodicity homomorphism to lie in $E_2^{d,0}$. In general we identify x with $[x]_2$.

Lemma 2. *Let G be a finite group having p-period d and let $X \to X/G$ be a principal covering. Suppose that $u \in H^d(G; \mathbf{Z}_p)$ generates the periodicity. Then $[u]_r$ exists for all $r \geq 2$, and*

$$- \cup [u]_r: E^{i,*} \to E^{i+d,*}$$

is an isomorphism for $i \geq r - 1 \geq 1$ *and an epimorphism for* $i \geq 0$, $r \geq 2$.

Remark. A version of this lemma has recently appeared in [2] for the case in which G is a cyclic group of order p. See also Skjelbred [10].

Proof of Lemma 2. By the hypothesis on the periodicity of G, the lemma is true for $r = 2$, since $\hat{H}^i(G; \mathbf{Z}_p) = H^i(G; \mathbf{Z}_p)$ for $i > 0$ and the map

$$H^0(G; \mathbf{Z}_p) \to \hat{H}^0(G; \mathbf{Z}_p)$$

is an epimorphism [1, Chapter XII]. Assume that it is true with $r - 1$ replacing r, and consider the following diagram:

$$\begin{array}{ccccc}
\operatorname{im}[d_{r-1}: E_{r-1}^{i-r+1,*+r-2} \to E_{r-1}^{i,*}] & \to & \ker[d_{r-1}: E_{r-1}^{i,*} \to E^{i+r-1,*-r+2}] & \to & E_r^{i,*} \to 0 \\
\big\downarrow \cup[u]_{r-1} \qquad \big\downarrow & & \big\downarrow \cup[u]_{r-1} & & \big\downarrow \cup[u]_r \\
\operatorname{im}[d_{r-1}: E_{r-1}^{i+d-r+1,*+r-2} \to E_{r-1}^{i+d,*}] & \to & \ker[d_{r-1}: E_{r-1}^{i+d,*} \to E_{r-1}^{i+d+r-1,*-r+2}] & \to & E_r^{i+d,*} \to 0
\end{array}$$

The conclusion follows by induction and the Five Lemma.

Lemma 1 follows directly from Lemma 3 below. Lemma 3 may be well known but for completeness we give a proof.

Lemma 3. *Let* G, *a finite group of p-period* d, *act freely on a path connected Hausdorff space* X *and suppose that*

(a) $H^i(X) = 0$ *for* $i > N$;

(b) $H^i(X/G) = 0$ *for* $i > K$;

(c) *If* $f: X/G \to BG$ *is the classifying map for the covering* $X \to X/G$ *then* $f^*(u) \neq 0$.

Then $H^i(X/G) = 0$ *for* $i > N$.

Remark. Since the following diagram commutes

$$\begin{array}{ccccc}
H^*(BG) & \xrightarrow{\cong} & H^*(G; H^0 X) = E_2^{*,0} & & \\
\big\downarrow & & \big\downarrow & & \\
H^*(X/G) & \longrightarrow & E_\infty^{*,0} & \longrightarrow & 0 \\
& & \big\downarrow & & \\
& & 0 & &
\end{array}$$

condition (c) of Lemma 3 is equivalent to the condition that $[u]_2$ persists to E_∞.

Proof of Lemma 3. Let $[x]_2 \in E_2^{st}$ be an infinite cycle, with $s + t > N$. Since $H^i(X/G) = 0$ for $i > K$, there is some i and some r such that

$[x]_r[u]_r^i \in E_r^{s+id,t}$ is a boundary. That is there is some $r \geq 2$ such that $[x]_r[u]_r^i = d_r[y]_r$, $[y]_r \in E^{s+id-r,t+r-1}$. Since we may as well assume that $[y]_2 \neq 0$, condition (a) of the lemma says that $t + r - 1 \leq N$, which together with $s + t > N$ yields $s + id - r \geq id$, so that by Lemma 2 $[y]_r = [z]_r[u]_r^i$ for some z. Then we have $[x]_r[u]_r^i = d_r([z]_r[u]_r^i) = (d_r[z]_r) \cdot [u]_r^i$, so by Lemma 2 $[x]_r$ is a boundary.

6. **An application of Lemma 2.** We wish to thank the referee for pointing out to us that Lemma 2 gives a simple proof of the main result in [7], which is:

Theorem. *Let G be a finite 2-group with maximal subgroup H. Then the sequence*

$$H^*(G; \mathbf{Z}_2) \xrightarrow{f} H^*(G; \mathbf{Z}_2) \xrightarrow{r} H^*(H; \mathbf{Z}_2)$$

is exact, where r is the restriction and f is cup product with $\pi^*(u)$, $u \in H^1(\mathbf{Z}_2; \mathbf{Z}_2)$ being the generator and $\pi: G \to G/H = \mathbf{Z}_2$ the projection.

Proof. Clearly $i: BH \to BG$ is a principal $\mathbf{Z}_2$-covering and $i^* = r$. Let F^s denote the standard decreasing filtration of $H^*(BG)$. Then $F^0H^*(BG) = H^*(BG)$ and the sequence

$$F^1H^*(BG) \to F^0H^*(BG) \xrightarrow{i^*} H^*(BH)$$

is exact. By Lemma 2, $F^1H^*(BG)$ is precisely the image of $H^*(BG)$ via cupping with the class $\pi^*(u)$. The result follows.

Note that the above method fails in the analogous situation for odd primes since in that case the periodicity class u lies in dimension 2.

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