COINCIDENCE POINT RESULTS FOR SPACES WITH FREE $\mathbb{Z}_p$ - ACTIONS

FRED COHEN AND EWING L. LUSK

ABSTRACT. Let $X$ support a free cyclic group action of prime order. We consider the question of determining when any map $f: X \to Y$ must identify two points of an orbit, and that of finding the minimum possible dimension of the union of such orbits when they exist.

1. Introduction. Let $X$ be a path-connected Hausdorff space which supports a free $\mathbb{Z}_p$ -action, where $\mathbb{Z}_p$ is the cyclic group of prime order $p$. Let $Y$ be any space and consider the question of whether there exists for any map $f: X \to Y$ a point $x \in X$ such that $f(x) = f(\sigma^i x)$ for some $i \neq 0$. The classical Borsuk-Ulam Theorem gives an affirmative answer for the case in which $p = 2$, $X = S^n$, and $Y = \mathbb{R}^n$. Various generalizations have appeared in [5], [9], [4], and [3].

We are particularly interested in the line of inquiry begun in [4], and present here some extensions of the above-mentioned results to the case in which $p$ is an arbitrary prime and $Y$ is a path-connected complex or manifold of dimension at least 2. (The assumptions on $Y$ can be weakened; they are present in order to facilitate explicit calculations.) Let $f: X \to Y$ and define

$$A(X, f) = \{ x \in X | f(\sigma^i x) = f(\sigma^j x) \text{ for some } i \neq j, 0 \leq i, j \leq p - 1 \}.$$ 

We will define a number $N$ depending on $Y$ and $p$ and prove the following two theorems. All cohomology is taken with $\mathbb{Z}_p$ coefficients.

We would like to thank the referee for pointing out the present (and much better) version of Lemma 2 and for the remarks which are contained in §6.

Theorem 1. If $X$ is a path-connected Hausdorff space and $H^i(X) = 0$ for $0 < i \leq N$, then $A(X, f) \neq \emptyset$.

Theorem 2. If $X$ is a closed $m$-manifold with $H^i(X) = 0$ for $0 \leq i \leq N$, then $\dim A(X, f) \geq m - N - 1$. 

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Remark. Here \( \dim A \) means the covering dimension of \( A \). By the argument which appears in [9], to prove \( \dim A \geq k \) it is sufficient to prove that \( \overline{H}^k(A) \neq 0 \), where \( \overline{H} \) denotes Alexander-Spanier cohomology.

We now define \( N \). Consider the subspace \( F(Y, p) = \{(y_1, \ldots, y_p) \mid y_i \neq y_j \text{ for } i \neq j\} \) of \( Y^p \). There is a free \( \pi_p \)-action on \( F(Y, p) \) given by cyclic permutation of coordinates. These spaces have been studied in [2]. If \( E \pi_p \) is a contractible free \( \pi_p \)-space, there is an equivariant map \( \phi \) inducing the following covering space maps:

\[
\begin{array}{ccc}
F(Y, p) & \xrightarrow{\phi} & F \pi_p \\
\downarrow & & \downarrow \\
F(Y, p)/\pi_p & \xrightarrow{\hat{\phi}} & B \pi_p
\end{array}
\]

The classifying space \( B \pi_p \) is known to have nonzero mod \( p \) cohomology in all dimensions. We define \( N = N(Y, p) \) to be the largest integer such that \( \hat{\phi}^* H^N(B \pi_p) \neq 0 \).

Some estimates on \( N \) for certain spaces \( Y \) are given in §3 below. In §4 are some examples which give bounds on possible improvements of the main theorems.

2. Proofs of Theorems 1 and 2. Define \( \psi: X \to Y^p \) by

\[
\psi(x) = (f(x), f(\sigma x), \ldots, f(\sigma^{p-1} x)).
\]

Proof of Theorem 1. If \( A(X, f) = \emptyset \), then \( \psi \) is an equivariant map of \( X \) into \( F(Y, p) \). Consider the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\psi} & F(Y, p) & \xrightarrow{\phi} & E \pi_p \\
\downarrow & & \downarrow & & \downarrow \\
X/\pi_p & \xrightarrow{\hat{\psi}} & F(Y, p)/\pi_p & \xrightarrow{\hat{\phi}} & B \pi_p
\end{array}
\]

Since \( H^i(X) = 0 \) for \( i \leq N \), it follows from the naturality of the spectral sequence for a covering (see [1]) that \( \hat{\phi} \psi \) is a monomorphism in degrees less than or equal to \( N + 1 \). This contradicts the fact that \( \hat{\phi}^* H^{N+1}(B \pi_p) = 0 \).

Proof of Theorem 2. Observe that \( \psi \) restricts to a map of \( X - A(X, f) \) into \( F(Y, p) \). We may assume that \( X - A(X, f) \) is path-connected. By the above argument, \( H^i(X - A(X, f)) \neq 0 \) for some \( 0 < j \leq N \), and hence \( H^j(X - A(X, f)) \neq 0 \) for some \( 0 < j \leq N \). By Alexander duality \( \overline{H}^{m-j}(X, A(X, f)) \neq 0 \). Similarly \( H^j(X) = 0 \) implies \( \overline{H}^{m-j}(X) = 0 \) and therefore by the cohomology exact sequence
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$$\overline{H}^{m-j-1}(A(X, f)) \neq 0.$$  

The result follows.

3. Estimation of $N$. In this section we give some estimates of $N$ for some spaces $Y$. The following lemma, whose proof is postponed until §5, is useful.

**Lemma 1.** If $H^i(F(Y, p)) = 0$ for $i \geq K$ and $H^1(F(Y, p)) = 0$, then $N(Y, p) < K$.

This means that an upper bound for $N(Y, p)$ can be found by finding the maximum dimension for which $H^i(F(Y, p)) \neq 0$. If $Y$ is a manifold this is easy to do, using the Serre spectral sequence for the fibration

$$F(Y - pt, j - 1) \to F(Y, j) \to Y.$$  

Some specific examples which have been calculated by this method are given in the table below. For each $X$ appearing there we give an $r$ such that if $X$ is any $r$-connected space supporting a free $\pi_p$-action generated by $\sigma$ then there is an $x \in X$ such that $f(x) = f(\sigma^i x)$ for some $i$.

<table>
<thead>
<tr>
<th>$Y$</th>
<th>$r(\geq N(Y, p))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^n$</td>
<td>$(n - 1)(p - 1)$</td>
</tr>
<tr>
<td>$S^n$</td>
<td>$(n - 1)(p - 1) + 1$</td>
</tr>
<tr>
<td>$S^n \times R^m$</td>
<td>$pm + (n - 1)(p - 1)$</td>
</tr>
<tr>
<td>$S^n \times S^m, m \geq n$</td>
<td>$pm + (n - 1)(p - 1) + 1$</td>
</tr>
</tbody>
</table>

Further, observe that $N(Y', p) \leq N(Y, p)$ if $Y'$ embeds in $Y$. Consequently

$$N(Y, p) \leq [ED(Y) - 1](p - 1),$$  

where $ED$ is the embedding dimension of $Y$. Hence one gets estimates of $N(Y, p)$ for spaces such as $RP^n$ and in some cases improved estimates for $S^n \times R^m$ and $S^n \times S^m$.

4. Examples. Here we give some examples to show that in some senses our results are the best possible for arbitrary $X$.

**Example 1.** Let $X = S^3 \times S^3$ with $\pi_2$-action $\sigma(x, y) = (-x, y)$. Define $f: S^3 \times S^3 \to S^3$ to be quaternionic multiplication. Then $f(x, y) \neq f(-x, y)$ for all $(x, y) \in S^3 \times S^3$, so $A(X, f) = \emptyset$ although $H^i(X) = 0$ for $0 < i < 12$ but $A(X, f) = \emptyset$. Consequently Theorem 1 is the best possible in the sense that $N$ cannot be replaced by $N - 1$.

**Example 2.** Define $f: S^3 \to R^2$ by $f(x_1, x_2, x_3, x_4) = (x_1, x_2)$ and let
have the antipodal \(\pi_2\)-action. It is easy to see that \(A(S^3, f) \cong S^1\) and \(N(R^2, 2) = 1\). Therefore Theorem 2 is the best possible in the sense that \(m - N - 1\) cannot be replaced by \(m - N\); i.e., \(N\) cannot be replaced by \(N - 1\) in the conclusion.

In case one puts more restrictive hypotheses on \(X\), then the results of Theorems 1 and 2 may not be the best possible. For instance if \(Y = RP^2\) and \(X = S^n\) with antipodal action, then it follows from [5] that \(\dim A \geq n - 2\), although Theorem 2 only gives \(\dim A \geq n - 4\).

For odd primes, it may be possible to improve the theorems, but not very much. The following example, with \(p = 3\), shows that \(N\) cannot be replaced by \(N - 2\) in Theorems 1 and 2.

Example 3. Regard \(S^3\) as \(\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_1|^2 + |z_2|^2 = 1\}\) and let \(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2\). Define \(\sigma : S^3 \to S^3\) by

\[
\sigma(z_1, z_2) = (e^{2\pi i/3}z_1, e^{2\pi i/3}z_2),
\]

and \(f : S^3 \to R^2\) by \(f(z_1, z_2) = (x_1, y_1)\). Then \(\sigma\) generates a \(\pi_3\)-action on \(S^3\) and

\[
A(S^3, f) = \{(x_1, y_1, x_2, y_2) \mid y_1 = -\sqrt{3}x_1, y_2 = -\sqrt{3}x_2\} \cong S^1.
\]

We have \(H^i(S^3) = 0\) for \(0 < i \leq N(R^2, 3)\), but the dimension of \(A\) is not greater than or equal to \(m - (N - 2) - 1\). If we define \(g : S^3 \to R^1\) by

\[
g(x_1, y_1, x_2, y_2) = (y_1 + \sqrt{3}x_1)^2 + (y_2 + \sqrt{3}x_2)^2
\]

and set

\[
h(x_1, y_1, x_2, y_2) = (f(x_1, y_1, x_2, y_2), g(x_1, y_1, x_2, y_2)),
\]

then \(h : S^3 \to R^3\) and \(A(S^3, h) = \emptyset\) although \(H^i(S^3) = 0\) for \(0 < i \leq 2 = N(R^3, 3) - 2\). Similar maps \(f : S^{2n-1} \to R^n\) and \(h : S^{2n-1} \to R^{n+1}\) can be constructed for larger \(p\) and \(n\). They show that in general \(N\) cannot be replaced by \(N - (n(p - 3) + 2)\).

Example 4. As in Example 3, regard \(S^3\) as \(\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_1|^2 + |z_2|^2 = 1\}\).

Define \(f : S^3 \to S^2\) to be the Hopf map, which takes a pair \((z_1, z_2)\) to its equivalence class under the relation \((z_1, z_2) \sim (z'_1, z'_2)\) if there is a \(r \in \mathbb{C}\) with \(|r| = 1\) such that \(z_1 = rz'_1, z_2 = rz'_2\). Let \(z_j = x_j + iy_j, j = 1, 2\), and define \(\lambda : S^3 \to S^3\) by \(\lambda(z_1, z_2) = (x_1 + ix_2, y_1 + iy_2)\). Let \(w = a + ib\) be any complex number with \(|a|^2 + |b|^2 = 1\) and \(b \neq 0\). Define \(\sigma : S^3 \to S^3\) by
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$\sigma(z_1, z_2) = (wz_1, \bar{w}z_2)$, and finally define $\rho: S^3 \to S^3$ to be $\lambda \sigma \lambda$. By evaluating the determinant of the appropriate $4 \times 4$ matrix, one can show that $\rho(z_1, z_2) \neq r \cdot (z_1, z_2)$ for all $(z_1, z_2) \in S^3$ and $r$ such that $|r| = 1$. For any odd prime $p$ we let $w = e^{2\pi i/p}$, so that $\rho$ generates a $\pi_p$-action. With this action on $S^3$ and this $f: S^3 \to S^2$, we have $A(S^3, f) = \emptyset$. (Of course for $p > 3$ we must apply the above calculation for powers of $w$ as well, but since $p$ is odd all powers have nonzero imaginary part, so the calculation goes through.) For $p = 3$ this example shows that $N$ cannot be replaced by $N - 1$ in Theorem 1.

Example 5. If $Y$ is a connected open 2-manifold, then by Corollary 3 of [4], whenever $\pi_1(X)$ is torsion the conclusion of Theorem 1 holds. Evidently Theorems 1 and 2 can be improved when $Y$ is a connected open 2-manifold.

5. Proof of Lemma 1. Our proof uses a lemma which may be of independent interest and which describes the differentials in the spectral sequence for a covering. We will present the argument in a more general setting than that required for Lemma 1.

Recall that a finite group $G$ is said to have $p$-period $d$ if $\hat{H}_i(G; A)$ and $\hat{H}^{i+d}(G; A)$ have isomorphic $p$-primary components for all $i$ and $A$, where $H^i(G; A)$ is the Tate cohomology of $G$ with coefficients in $A$. We assume that the periodicity homomorphism

$$\phi: \hat{H}_i(G; A) \to \hat{H}^{i+d}(G; A)$$

is given by $\phi(m) = m \cup u$ for $u$ a fixed element in $\hat{H}^d(G; \mathbb{Z}_p)$, where "$\cup$" denotes the internal cup product and $G$ acts trivially on $\mathbb{Z}_p$. Some examples are $\mathbb{Z}_p$, in which case $d = 2$ [8], and all subgroups of $\Sigma_j$, the symmetric group on $j$ letters, for $j < 2p$ [11].

If $G$ acts freely on $X$, there is a spectral sequence for the covering $X \to X/G$, which is a spectral sequence of algebras, converging to $H^*(X/G)$ with $E_{2}^{**} = H^\ast(G, H^\ast(X))$. Let $[x]_r$ denote a class in $E_r$ which is represented by the class $[x]_2$ in $E_2$ (of course $[x]_r$ may not be defined for all $[x]_2$ in $E_2$). Observe that we may consider the class $u$ in $H^d(G; \mathbb{Z}_p)$ which induces the periodicity homomorphism to lie in $E_{2,0}^{d,0}$. In general we identify $x$ with $[x]_2$.

Lemma 2. Let $G$ be a finite group having $p$-period $d$ and let $X \to X/G$ be a principal covering. Suppose that $u \in H^d(G; \mathbb{Z}_p)$ generates the periodicity. Then $[u]_r$ exists for all $r \geq 2$, and

$$- \bigcup [u]_r: E_{i,*}^{r} \to E_{i+d,*}^{r}$$
is an isomorphism for \( i \geq r - 1 \geq 1 \) and an epimorphism for \( i \geq 0, \ r \geq 2 \).

Remark. A version of this lemma has recently appeared in [2] for the case in which \( G \) is a cyclic group of order \( p \). See also Skjelbred [10].

Proof of Lemma 2. By the hypothesis on the periodicity of \( G \), the lemma is true for \( r = 2 \), since \( \hat{H}^i(G; \mathbb{Z}_p) = H^i(G; \mathbb{Z}_p) \) for \( i > 0 \) and the map
\[
H^0(G; \mathbb{Z}_p) \to \hat{H}^0(G; \mathbb{Z}_p)
\]
is an epimorphism [1, Chapter XII]. Assume that it is true with \( r - 1 \) replacing \( r \), and consider the following diagram:

![Diagram](image)

The conclusion follows by induction and the Five Lemma.

Lemma 1 follows directly from Lemma 3 below. Lemma 3 may be well known but for completeness we give a proof.

Lemma 3. Let \( G \), a finite group of \( p \)-period \( d \), act freely on a path connected Hausdorff space \( X \) and suppose that
(a) \( \tau_i(X) = 0 \) for \( i > N \);
(b) \( H^i(X/G) = 0 \) for \( i > K \);
(c) \( f: X/G \to BG \) is the classifying map for the covering \( X \to X/G \) then \( f^*(u) \neq 0 \).

Then \( H^i(X/G) = 0 \) for \( i > N \).

Remark. Since the following diagram commutes
\[
\begin{array}{ccc}
H^*(BG) & \xrightarrow{\cong} & H^*(G; H^0X) = E^*_2,0 \\
\downarrow & & \downarrow \\
H^*(X/G) & \to & E^*_\infty,0 \\
& \downarrow & \\
& 0 & \\
\end{array}
\]
condition (c) of Lemma 3 is equivalent to the condition that \([u]_2\) persists to \( E_\infty \).

Proof of Lemma 3. Let \([x]_2 \in E^*_2\) be an infinite cycle, with \( s + t > N \). Since \( H^i(X/G) = 0 \) for \( i > K \), there is some \( i \) and some \( r \) such that
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$[x][u]^i \in E^{s+id,t}$ is a boundary. That is there is some $r \geq 2$ such that $[x][u]^i = d_r[y]^i, [y]^i \in E^{s+id-r,t+r-1}$. Since we may as well assume that $[y]^i \neq 0$, condition (a) of the lemma says that $t + r - 1 \leq N$, which together with $s + t > N$ yields $s + id - r \geq id$, so that by Lemma 2 $[y]^i = [z][u]^i$ for some $z$. Then we have $[x][u]^i = d_r([z][u]^i) = (d_r[z]^i) \cdot [u]^i$, so by Lemma 2 $[x]^i$ is a boundary.

6. An application of Lemma 2. We wish to thank the referee for pointing out to us that Lemma 2 gives a simple proof of the main result in [7], which is:

**Theorem.** Let $G$ be a finite 2-group with maximal subgroup $H$. Then the sequence

$$H^*(G; \mathbb{Z}_2) \xrightarrow{f} H^*(G; \mathbb{Z}_2) \xrightarrow{r} H^*(H; \mathbb{Z}_2)$$

is exact, where $r$ is the restriction and $f$ is cup product with $\pi^*(u), u \in H^1(\mathbb{Z}_2; \mathbb{Z}_2)$ being the generator and $\pi: G \to G/H = \mathbb{Z}_2$ the projection.

**Proof.** Clearly $i: BH \to BG$ is a principal $\mathbb{Z}_2$-covering and $i^* = r$. Let $F^s$ denote the standard decreasing filtration of $H^*(BG)$. Then $F^0H^*(BG) = H^*(BG)$ and the sequence

$$F^1H^*(BG) \to F^0H^*(BG) \xrightarrow{i^*} H^*(BH)$$

is exact. By Lemma 2, $F^1H^*(BG)$ is precisely the image of $H^*(BG)$ via cupping with the class $\pi^*(u)$. The result follows.

Note that the above method fails in the analogous situation for odd primes since in that case the periodicity class $u$ lies in dimension 2.

REFERENCES


DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115