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# VANISHING HOMOLOGY OVER NILPOTENT GROUPS[1]

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ABSTRACT. Let $\pi$ be a nilpotent group and let $M$ be a $\pi$-module. Under certain finiteness assumptions we prove that the twisted homology groups $H_i(\pi, M)$ vanish for all positive $i$ whenever $H_0(\pi, M) = 0$.

The purpose of this note is to prove the following vanishing theorem:

(1) **Theorem.** *Let $\pi$ be a finitely generated nilpotent group, and let $M$ be a $\pi$-module which is finitely generated over $\mathbf{Z}[\pi]$. Assume that $H_0(\pi, M) = 0$. Then $H_i(\pi, M) = 0$ for all $i \geq 0$.*

In the statement of this theorem, the $\pi$-module $M$ is, as usual, an abelian group equipped with a left $\pi$-action, $\mathbf{Z}[\pi]$ is the integral group ring of $\pi$, and $H_i(\pi, M)$, $i \geq 0$, are the twisted homology groups defined in [4].

With a little care, the proof below will also yield the following more general result:

(2) **Theorem.** *Let $\pi$ be as in* (1), *and let $\{M_s\}_{s\geq 0}$ be a tower of $\pi$-modules, each of which is finitely generated over $\mathbf{Z}[\pi]$. Then if $\{H_0(\pi, M_s)\}_{s\geq 0}$ is protrivial, all the other towers $\{H_i(\pi, M_s)\}_{s\geq 0}$ for $i > 0$ are protrivial too.*

For a definition of the terms in this statement, and a discussion of the basic properties of towers, see [1].

Several topological applications of these theorems will be examined in forthcoming papers [2], [3]. Our statements are parallel to the results of [5], although very different in detail.

The author does not know in what sense these theorems are "best possible." There are examples to show that the finite generation condition on $M$ and the nilpotency condition on $\pi$ are unavoidable, but the finite generation condition on $\pi$ may well be redundant. In fact, a simple proof of (1)

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for arbitrary abelian $\pi$ appears at the end of this note, but the author cannot see how to generalize it.

**Preliminaries.** In the discussion below, a $\pi$-module $M$ such that $H_0(\pi, M)$ vanishes is called *perfect*; a module $M$ such that $H_i(\pi, M)$ vanishes for all $i \geq 0$ is called *acyclic*. Three observations will be used repeatedly:

(3) *Any quotient module of a perfect $\pi$-module is perfect.*

This follows at once from the fact that $H_0(\pi, -)$ is right exact.

(4) *If $M'$ is a submodule of $M$, $M$ is perfect, and $M/M'$ is acyclic, then $M'$ is perfect.*

This follows at once from the long exact homology sequence of $0 \to M' \to M \to M/M' \to 0$.

(5) *The integral group ring of a finitely generated nilpotent group is (left and right) noetherian.*

This is the backbone of the argument below. A proof is indicated in [6].

Now the proof of (1) proceeds by induction on the number of central cyclic extensions needed to construct $\pi$. Consequently, we can assume that $\sigma$ is a cyclic subgroup of the center of $\pi$ and that the obvious inductive theorem is known for $\pi/\sigma$-modules. Let $M$ be a finitely generated perfect $\mathbf{Z}[\pi]$-module. We let $s$ be a generator of $\sigma$, and $T$ the endomorphism of $M$ given by $m \mapsto m - s \cdot m$ for all $m \in M$. The letter $p$ will denote the order of $\sigma$, which can be assumed either prime or infinite; if $p$ is infinity, then by convention every element of every abelian group is said to be of order $p$.

**Special cases.** In this paragraph we will prove that $M$ is acyclic if it has one of the following three special forms:

**Type I.** $T$ is injective on $M$.

**Type II.** $T$ is the zero map $M \to M$, and $M$ has no elements of order $p$. (This type is trivial if $p = \infty$.)

**Type III.** $T$ is the zero map $M \to M$, and every element of $M$ has order $p$.

If $M$ falls into one of these three classes, we will compute the $E^2$-term of the Lyndon spectral sequence [M]

$$E^2_{p,q} = H_p(\pi/\sigma, H_q(\sigma, M)) \Rightarrow H_{p+q}(\pi, M)$$

and show that it vanishes. Actually, the computation below will only show that

$$H_0(\pi/\sigma, H_j(\sigma, M)) = E^2_{0,j} = 0 \quad \text{for all } j.$$

The induction hypothesis, together with the easily proven fact that each $H_j(\sigma, M)$ is a finitely generated $\mathbf{Z}[\pi/\sigma]$ -module, will then give $H_i(\pi/\sigma, H_j(\sigma, M)) = E^2_{i,j} = 0$ for all $i, j \geq 0$.

The computation depends on explicit knowledge of the homology groups $H_i(\sigma, M)$, which are well known to be given as follows [4]:

*Case* I. *$\sigma$ is finite of order $p$.* Let $N$ be the endomorphism of $M$ given by $1 + s + s^2 + \cdots + s^{p-1}$.

$$H_0(\sigma, M) = M/\text{image}(T),$$

$$H_{2i}(\sigma, M) = \text{kernel}(N)/\text{image}(T), \qquad i \geq 1,$$

$$H_{2i+1}(\sigma, M) = \text{kernel}(T)/\text{image}(N), \qquad i > 0.$$

*Case* II. *$\sigma$ is infinite cyclic.*

$$H_0(\sigma, M) = M/\text{image}(T), \quad H_1(\sigma, M) = \text{kernel}(T), \quad H_i(\sigma, M) = 0, \qquad i > 1.$$

The computation now breaks into three parts, according to the structure of $M$. Recall that we are given $H_0(\pi/\sigma, H_0(\sigma, M)) \simeq H_0(\pi, M) = 0$.

**Type I.** In this case $H_{2i+1}(\sigma, M) = 0$ for all $i \geq 0$. If $\sigma$ is infinite cyclic, $H_{2i}(\sigma, M)$, $i \geq 1$, is zero, so there is nothing more to show. Otherwise, if $\sigma$ is finite, each $H_{2i}(\sigma, M)$, $i \geq 1$, is a sub-$\pi/\sigma$-module of $H_0(\sigma, M)$, and so, by the argument of (4), $H_0(\pi/\sigma, H_{2i}(\sigma, M)) = 0$.

**Type II.** If $\sigma$ is infinite, there is nothing to show. If $\sigma$ is finite, then $H_{2i}(\sigma, M) = 0$ for $i \geq 1$ and $H_{2i+1}(\sigma, M)$, $i \geq 0$, is a quotient $\pi/\sigma$-module of $H_0(\sigma, M)$, and so, by (3), $H_0(\pi/\sigma, H_{2i+1}(\sigma, M)) = 0$.

**Type III.** If $\sigma$ is finite, then $H_i(\sigma, M) \simeq M \simeq H_0(\sigma, M)$ for all $i > 0$, so that $H_0(\pi/\sigma, H_i(\sigma, M)) = 0$. If $\sigma$ is infinite, then $H_1(\sigma, M) \simeq M \simeq H_0(\sigma, M)$, and $H_i(\sigma, M) = 0$ for $i > 1$, so the same argument works.

**The general case.** Let $M$ be an arbitrary perfect finitely generated $\mathbf{Z}[\pi]$-module. In order to show that $M$ is acyclic, it is enough to show that there is a finite $\pi$-filtration of $M$,

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k = M$$

such that each filtration quotient $F_{i+1}/F_i$ is of Type I, II, or III. Indeed, if such a filtration exists, then we show by descending induction on $i$ that

$M/F_i$ is acyclic for all $0 \le i \le k$. The induction starts with the fact that $M/F_k = M/M = 0$ is acyclic. Now suppose that $M/F_i$ is acyclic, $i > 0$. There is a short exact sequence

$$0 \to F_i/F_{i-1} \to M/F_{i-1} \to M/F_i \to 0.$$

By (3) $M/F_{i-1}$, as a quotient of $M$, is a perfect $\pi$-module. The long exact homology sequence of this short exact sequence, and the fact that $M/F_i$ is acyclic, show that $F_i/F_{i-1}$ is perfect. Since this $\pi$-module is of Type I, II, or III, it must be acyclic. Another look at the long exact homology sequence verifies that $M/F_{i-1}$ is acyclic.

Constructing such a filtration of $M$ is not hard. Let $T$ be as above, and define $F_i$ $(i \ge 0)$ by

$$F_0 = \{0\} \subset M, \quad F_i = \operatorname{kernel}\{T^i : M \to M\}, \qquad i \ge 1.$$

Since $\sigma$ is in the center of $\pi$, $\{F_i | i \ge 0\}$ is a family of $\pi$-equivariant submodules of $M$. By the noetherian condition (5), this family must have a maximal element, so, for some $K \ge 0$, $F_K = F_{K+1}$. Then $M/F_K$ is of Type I. If $\sigma$ is infinite cyclic, each $F_{i+1}/F_i$ is already of Type III, and we are done. Otherwise, if $\sigma$ is finite of order $p$, it is enough to show that each $F_{i+1}/F_i$ can be filtered (in a $\pi$-equivariant way) so that the filtration quotients are of Type II or III. To do this, pick $0 \le i \le K-1$, and define $G_j$ $(j \ge 0)$ by

$$G_0 = F_i, \qquad G_j = \{x \in F_{i+1} | p^j x \in F_i\}.$$

Again by the noetherian condition, there is some $J \ge 0$ such that $G_J = G_{J+1}$. Clearly $F_{i+1}/G_J$ is of Type II, and each $G_{j+1}/G_j$ is of Type III. This completes the proof.

**A simple proof of the abelian case.** We give a conceptual proof for arbitrary abelian $\pi$ that any perfect $\pi$-module $M$ which is finitely generated over $\mathbf{Z}[\pi]$ is acyclic. It is enough to assume that $M$ has a single generator $m$ over $\mathbf{Z}[\pi]$; induction on the number of generators, using (4), then gives the general case.

Let $I \subseteq \mathbf{Z}[\pi]$ be the augmentation ideal—the kernel of the natural epimorphism $\mathbf{Z}[\pi] \to \mathbf{Z}$. By hypothesis, $H_0(\pi, M) = \mathbf{Z} \otimes_{\mathbf{Z}[\pi]} M = M/I \cdot M = 0$, so there must be some $r \in I$ such that $r \cdot m = m$. Since $\pi$ is abelian, left multiplication by $r$ commutes with the action of $\pi$ and so must induce the identity map $M \to M$, and therefore the identity map $H_*(\pi, M) \to H_*(\pi, M)$.

However, again since $\pi$ is abelian, left multiplication by $r$ is easily seen to induce a natural transformation of the functor $H_*(\pi, -)$ into itself; this natural transformation is evidently zero on $H_0(\pi, -)$ and so, by the basic theorems about derived functors, must be identically zero. Consequently, the identity map $H_*(\pi, M) \to H_*(\pi, M)$ coincides with the zero map.

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