VANISHING HOMOLOGY OVER NILPOTENT GROUPS

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ABSTRACT. Let $\pi$ be a nilpotent group and let $M$ be a $\pi$-module.
Under certain finiteness assumptions we prove that the twisted homology
groups $H_i(\pi, M)$ vanish for all positive $i$ whenever $H_0(\pi, M) = 0$.

The purpose of this note is to prove the following vanishing theorem:

(1) Theorem. Let $\pi$ be a finitely generated nilpotent group, and let $M$
be a $\pi$-module which is finitely generated over $\mathbb{Z}[\pi]$. Assume that $H_0(\pi, M) = 0$. Then $H_i(\pi, M) = 0$ for all $i \geq 0$.

In the statement of this theorem, the $\pi$-module $M$ is, as usual, an abelian group equipped with a left $\pi$-action, $\mathbb{Z}[\pi]$ is the integral group ring of $\pi$, and $H_i(\pi, M)$, $i \geq 0$, are the twisted homology groups defined in [4].

With a little care, the proof below will also yield the following more general result:

(2) Theorem. Let $\pi$ be as in (1), and let $\{M_s\}_{s \geq 0}$ be a tower of $\pi$-modules, each of which is finitely generated over $\mathbb{Z}[\pi]$. Then if $\{H_0(\pi, M_s)\}_{s \geq 0}$ is protrivial, all the other towers $\{H_i(\pi, M_s)\}_{s \geq 0}$ for $i > 0$ are protrivial too.

For a definition of the terms in this statement, and a discussion of the basic properties of towers, see [1].

Several topological applications of these theorems will be examined in forthcoming papers [2], [3]. Our statements are parallel to the results of [5], although very different in detail.

The author does not know in what sense these theorems are "best possible." There are examples to show that the finite generation condition on $M$ and the nilpotency condition on $\pi$ are unavoidable, but the finite generation condition on $\pi$ may well be redundant. In fact, a simple proof of (1)
Preliminaries. In the discussion below, a \( \pi \)-module \( M \) such that \( H_0(\pi, M) \) vanishes is called \emph{perfect}; a module \( M \) such that \( H_i(\pi, M) \) vanishes for all \( i \geq 0 \) is called \emph{acyclic}. Three observations will be used repeatedly:

1. \textit{Any quotient module of a perfect \( \pi \)-module is perfect.}

   This follows at once from the fact that \( H_0(\pi, -) \) is right exact.

2. \textit{If \( M' \) is a submodule of \( M \), \( M \) is perfect, and \( M/M' \) is acyclic, then \( M' \) is perfect.}

   This follows at once from the long exact homology sequence of \( 0 \rightarrow M' \rightarrow M \rightarrow M/M' \rightarrow 0. \)

3. \textit{The integral group ring of a finitely generated nilpotent group is (left and right) noetherian.}

   This is the backbone of the argument below. A proof is indicated in [6].

Now the proof of (1) proceeds by induction on the number of central cyclic extensions needed to construct \( \pi \). Consequently, we can assume that \( \sigma \) is a cyclic subgroup of the center of \( \pi \) and that the obvious inductive theorem is known for \( \pi/\sigma \)-modules. Let \( M \) be a finitely generated perfect \( \mathbb{Z}[\pi] \)-module. We let \( s \) be a generator of \( \sigma \), and \( T \) the endomorphism of \( M \) given by \( m \mapsto m - s \cdot m \) for all \( m \in M \). The letter \( p \) will denote the order of \( \sigma \), which can be assumed either prime or infinite; if \( p \) is infinity, then by convention every element of every abelian group is said to be of order \( p \).

Special cases. In this paragraph we will prove that \( M \) is acyclic if it has one of the following three special forms:

- **Type I.** \( T \) is injective on \( M \).
- **Type II.** \( T \) is the zero map \( M \rightarrow M \), and \( M \) has no elements of order \( p \).
  (This type is trivial if \( p = \infty \).)
- **Type III.** \( T \) is the zero map \( M \rightarrow M \), and every element of \( M \) has order \( p \).

If \( M \) falls into one of these three classes, we will compute the \( E^2 \)-term of the Lyndon spectral sequence [M]

\[
E^2_{p, q} = H_p(\pi/\sigma, H_q(\sigma, M)) \Rightarrow H_{p+q}(\pi, M)
\]

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and show that it vanishes. Actually, the computation below will only show that
\[ H_0(\pi/\alpha, H_j(\sigma, M)) = E_{0,j}^2 = 0 \quad \text{for all } j. \]

The induction hypothesis, together with the easily proven fact that each
\( H_j(\sigma, M) \) is a finitely generated \( \mathbb{Z}[\pi/\alpha] \)-module, will then give
\[ H_0(\pi/\alpha, H_j(\sigma, M)) = E_{0,j}^2 = 0 \quad \text{for all } i, j \geq 0. \]

The computation depends on explicit knowledge of the homology groups
\( H_i(\sigma, M) \), which are well known to be given as follows [4]:

**Case I.** \( \sigma \) is finite of order \( p \). Let \( N \) be the endomorphism of \( M \) given by \( 1 + s + s^2 + \cdots + s^{p-1} \).

\[ H_0(\sigma, M) = M/\text{image}(T), \]
\[ H_{2i}(\sigma, M) = \text{kernel}(N)/\text{image}(T), \quad i \geq 1, \]
\[ H_{2i+1}(\sigma, M) = \text{kernel}(T)/\text{image}(N), \quad i > 0. \]

**Case II.** \( \sigma \) is infinite cyclic.

\[ H_0(\sigma, M) = M/\text{image}(T), \quad H_1(\sigma, M) = \text{kernel}(T), \quad H_i(\sigma, M) = 0, \quad i > 1. \]

The computation now breaks into three parts, according to the structure of \( M \). Recall that we are given \( H_0(\pi/\alpha, H_j(\sigma, M)) \simeq H_0(\pi, M) = 0. \)

**Type I.** In this case \( H_{2i+1}(\sigma, M) = 0 \) for all \( i \geq 0 \). If \( \sigma \) is infinite cyclic, \( H_{2i}(\sigma, M), i \geq 1 \), is zero, so there is nothing more to show. Otherwise, if \( \sigma \) is finite, each \( H_{2i}(\sigma, M), i \geq 1 \), is a sub-\( \pi/\alpha \)-module of \( H_0(\sigma, M) \), and so, by the argument of (4), \( H_0(\pi/\alpha, H_{2i}(\sigma, M)) = 0. \)

**Type II.** If \( \sigma \) is infinite, there is nothing to show. If \( \sigma \) is finite, then \( H_{2i}(\sigma, M) = 0 \) for \( i \geq 1 \) and \( H_{2i+1}(\sigma, M), i \geq 0 \), is a quotient \( \pi/\alpha \)-module of \( H_0(\sigma, M) \), and so, by (3), \( H_0(\pi/\alpha, H_{2i+1}(\sigma, M)) = 0. \)

**Type III.** If \( \sigma \) is finite, then \( H_i(\sigma, M) \cong M \cong H_0(\sigma, M) \) for all \( i > 0 \), so that \( H_0(\pi/\alpha, H_i(\sigma, M)) = 0. \) If \( \sigma \) is infinite, then \( H_1(\sigma, M) \cong M \cong H_0(\sigma, M) \), and \( H_i(\sigma, M) = 0 \) for \( i > 1 \), so the same argument works.

**The general case.** Let \( M \) be an arbitrary perfect finitely generated \( \mathbb{Z}[\pi] \)-module. In order to show that \( M \) is acyclic, it is enough to show that there is a finite \( \pi \)-filtration of \( M \),
\[ 0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k = M \]
such that each filtration quotient \( F_{i+1}/F_i \) is of Type I, II, or III. Indeed, if such a filtration exists, then we show, by descending induction on \( i \) that
$M/F_i$ is acyclic for all $0 \leq i \leq k$. The induction starts with the fact that $M/F_k = M/M = 0$ is acyclic. Now suppose that $M/F_i$ is acyclic, $i > 0$. There is a short exact sequence

$$0 \rightarrow F_i/F_{i-1} \rightarrow M/F_{i-1} \rightarrow M/F_i \rightarrow 0.$$ 

By (3) $M/F_{i-1}$, as a quotient of $M$, is a perfect $\pi$-module. The long exact homology sequence of this short exact sequence, and the fact that $M/F_i$ is acyclic, show that $F_i/F_{i-1}$ is perfect. Since this $\pi$-module is of Type I, II, or III, it must be acyclic. Another look at the long exact homology sequence verifies that $M/F_{i-1}$ is acyclic.

Constructing such a filtration of $M$ is not hard. Let $T$ be as above, and define $F_i$ ($i \geq 0$) by

$$F_0 = \{0\} \subset M, \quad F_i = \ker\{T^i : M \rightarrow M\}, \quad i \geq 1.$$ 

Since $\sigma$ is in the center of $\pi$, $\{F_i | i \geq 0\}$ is a family of $\pi$-equivariant submodules of $M$. By the noetherian condition (5), this family must have a maximal element, so, for some $K \geq 0$, $F_K = F_{K+1}$. Then $M/F_K$ is of Type I. If $\sigma$ is infinite cyclic, each $F_{i+1}/F_i$ is already of Type III, and we are done. Otherwise, if $\sigma$ is finite of order $p$, it is enough to show that each $F_{i+1}/F_i$ can be filtered (in a $\pi$-equivariant way) so that the filtration quotients are of Type II or III. To do this, pick $0 \leq i \leq K - 1$, and define $G_j$ ($j \geq 0$) by

$$G_0 = F_i, \quad G_j = \{x \in F_{i+1} | p^jx \in F_i\}.$$ 

Again by the noetherian condition, there is some $J \geq 0$ such that $G_J = G_{J+1}$. Clearly $F_{i+1}/G_J$ is of Type II, and each $G_{j+1}/G_j$ is of Type III. This completes the proof.

A simple proof of the abelian case. We give a conceptual proof for arbitrary abelian $\pi$ that any perfect $\pi$-module $M$ which is finitely generated over $\mathbb{Z}[\pi]$ is acyclic. It is enough to assume that $M$ has a single generator $m$ over $\mathbb{Z}[\pi]$; induction on the number of generators, using (4), then gives the general case.

Let $I \subseteq \mathbb{Z}[\pi]$ be the augmentation ideal—the kernel of the natural epimorphism $\mathbb{Z}[\pi] \rightarrow \mathbb{Z}$. By hypothesis, $H_0(\pi, M) = \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} M = M/I \cdot M = 0$, so there must be some $r \in I$ such that $r \cdot m = m$. Since $\pi$ is abelian, left multiplication by $r$ commutes with the action of $\pi$ and so must induce the identity map $M \rightarrow M$, and therefore the identity map $H_*(\pi, M) \rightarrow H_*(\pi, M)$. 

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However, again since $\pi$ is abelian, left multiplication by $r$ is easily seen to induce a natural transformation of the functor $H_*(\pi, -)$ into itself; this natural transformation is evidently zero on $H_0(\pi, -)$ and so, by the basic theorems about derived functors, must be identically zero. Consequently, the identity map $H_*(\pi, M) \to H_*(\pi, M)$ coincides with the zero map.

REFERENCES