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TOEPLITZ OPERATORS ASSOCIATED WITH ISOMETRIES

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ABSTRACT. Shift analysis includes abstract treatments of inner-outer factorization problems, the factorization problem for nonnegative functions on a circle, and Szegö's infimum problem (for scalar or operator valued functions). These problems are here generalized to a setting where the shift operator is replaced by a pair of isometries.

1. **Introduction.** The inner-outer factorization problem for bounded analytic functions on a disk, the factorization problem for nonnegative functions on a circle, and Szegö's infimum problem are known to be equivalent to factorization and extremal problems for analytic and Toeplitz operators, i.e. operators A and T on a Hilbert space $\mathcal{H}$ which satisfy identities $AS = SA$ and $S^*TS = T$, where S is a unilateral shift operator on $\mathcal{H}$. In this paper we indicate what modifications are needed to generalize these problems to operators A and T, $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_1)$, which satisfy identities $AV_1 = V_2A$ and $V_1^*TV_1 = T$, where V_1 and V_2 are isometries acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We obtain an inner-outer factorization theorem, a weak form of Beurling's theorem for isometries, characterizations of products AA^* and A^*A, and a generalized form of Szegö's infimum for T.

The treatment of these problems for the shift case can be found in Moore [3] and Rosenblum and Rovnyak [7] (other sources are cited in [3], [7]). Generalizations of the kind we consider were first obtained by Page [5], [6]. For related results see Devinatz and Shinbrot [1], Douglas [2], and Muhly [4].

2. **Notation and terminology.** If V is an isometry on a Hilbert space $\mathcal{H}$, the Wold decomposition of V is written $V = S \oplus U$, $\mathcal{H} = \mathcal{K} \oplus \mathcal{L}$, where

$$\mathcal{K} = \sum_{k=0}^{\infty} \oplus V^k\mathcal{C}, \quad \mathcal{C} = \ker V^*, \quad \mathcal{L} = \bigcap_{k=0}^{\infty} V^k\mathcal{H}.$$

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Thus $S = V|\mathcal{K}$ and $U = V|\mathcal{L}$ are the shift and unitary parts of V respectively. The same notation is used with subscripts 1, 2, 3, T, $\mathfrak{M}$ in different parts of the paper without further explanation.

Let V_1 and V_2 be isometries acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $T \in \mathcal{B}(\mathcal{H}_1)$. Then A is called

(i) (V_1, V_2)-*analytic* if $AV_1 = V_2A$,

(ii) (V_1, V_2)-*inner* if A is partially isometric and (V_1, V_2)-analytic,

and

(iii) (V_1, V_2)-*outer* if A is (V_1, V_2)-analytic and $(A\mathcal{H}_1)^-$ reduces V_2.

We say that T is V_1-*Toeplitz* if $V_1^*TV_1 = T$.

Let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2)$. We write $B \prec A$ if B is unitarily equivalent to the restriction of A to some closed invariant subspace of A. It is easy to see that $B \prec A$ if and only if there is an isometry $W \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $WB = AW$.

3. **Products AA^* and invariant subspaces.** In this section V_1 and V_2 denote fixed isometries acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

Theorem 1. *Let R be a nonnegative operator on $\mathcal{H}_2$. Then $R = AA^*$ for some (V_1, V_2)-analytic operator A if and only if*

(a) $R - V_2RV_2^* = J^*J$ *for some* $J \in \mathcal{B}(\mathcal{H}_2, \mathcal{C}_1)$, *and*

(b) $V_2|(Q\mathcal{H}_2)^- \prec U_1$ *where* $Q = \text{s-lim}_{n\to\infty} V_2^nRV_2^{*n}$.

If $U_2 \prec U_1$, then (b) holds automatically. The existence of the limit in (b) is implied by (a) because

$$V_2^nRV_2^{*n} = V_2^{n+1}RV_2^{*n+1} + V_2^n(R - V_2RV_2^*)V_2^{*n} \geq V_2^{n+1}RV_2^{*n+1}$$

for all $n = 1, 2, 3, \cdots$.

Proof. Assume $R = AA^*$ where A is (V_1, V_2)-analytic. Then $R - V_2RV_2^* = A(I - V_1V_1^*)A^*$, and since $I - V_1V_1^*$ is the projection of $\mathcal{H}_1$ on $\mathcal{C}_1$, we obtain (a) with $J = (I - V_1V_1^*)A^*$. Now

$$Q = \text{s-lim}_{n\to\infty} AV_1^nV_1^{*n}A^* = AEA^*$$

where E is the projection of $\mathcal{H}_1$ on $\mathcal{L}_1$. Let $W: (Q\mathcal{H}_2)^- \to \mathcal{L}_1$ be the unique isometry such that $W: Q^{1/2}f \to EA^*f$, $f \in \mathcal{H}_2$. It can be shown that $(Q\mathcal{H}_2)^-$ is invariant under V_2, and $WV_2g = U_1Wg$, $g \in Q^{1/2}\mathcal{H}_2$. Then (b) follows.

Conversely, let (a) and (b) hold. Then $V_2QV_2^* = Q \geq 0$. By (b) there is an isometry $W: (Q\mathcal{H}_2)^- \to \mathcal{L}_1$ such that $WV_2g = U_1Wg$, $g \in (Q\mathcal{H}_2)^-$. Then

$WV_2Q^{1/2} = V_1WQ^{1/2}$. But V_2 commutes with $Q^{1/2}$, so $(WQ^{1/2})V_2 = V_1(WQ^{1/2})$. Hence $V_1^*(WQ^{1/2})V_2V_2^* = (WQ^{1/2})V_2^*$, and since $Q\mathcal{K}_2 = (0)$,

$$V_1^*(WQ^{1/2}) = (WQ^{1/2})V_2^*.$$

Therefore the operator $M = (WQ^{1/2})^*$ is (V_1, V_2)-analytic.

On iterating the identity in (a) we obtain

$$R = V_2^{n+1}RV_2^{*n+1} + \sum_{j=0}^{n} V_2^jJ^*JV_2^{*j}, \qquad n \geq 0;$$

hence

$$R = Q + \sum_{j=0}^{\infty} V_2^jJ^*JV_2^{*j}$$

with strong convergence of the series. Define $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ by

$$A^* = M^* + \sum_{j=0}^{\infty} V_1^jJV_2^{*j}.$$

The series converges strongly, and for any $f \in \mathcal{H}_2$,

$$\|A^*f\|^2 = \|M^*f\|^2 + \sum_{j=0}^{\infty} \|V_1^jJV_2^{*j}f\|^2$$

$$= \left\langle \left[Q + \sum_{j=0}^{\infty} V_2^jJ^*JV_2^{*j}\right]f, f \right\rangle = \langle Rf, f\rangle.$$

Hence $R = AA^*$. A straightforward argument shows that A is (V_1, V_2)-analytic.

Theorem 2. *Let $\mathfrak{M}$ be a closed invariant subspace of V_2, and let $V_{\mathfrak{M}} = V_2|\mathfrak{M}$. Then for $\mathfrak{M}$ to have the form $\mathfrak{M} = B\mathcal{H}_1$ where B is (V_1, V_2)-inner it is necessary and sufficient that $S_{\mathfrak{M}} \prec S_1$ and $U_{\mathfrak{M}} \prec U_1$.*

No stronger theorem can be proved, because each of the four true-false possibilities for the relations $S_{\mathfrak{M}} \prec S_1$ and $U_{\mathfrak{M}} \prec U_1$ can be realized in examples. Note that if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $V_1 = V_2 = V$, then the relation $U_{\mathfrak{M}} \prec U_1$ holds automatically. If further V has no unitary part, then the relation $S_{\mathfrak{M}} \prec S_1$ is automatic by a well-known property of shift operators.

Proof. Let R be the projection of $\mathcal{H}_2$ on $\mathfrak{M}$. Then $\mathfrak{M} = B\mathcal{H}_1$, where B is (V_1, V_2)-inner, if and only if $R = AA^*$, where A is (V_1, V_2)-analytic. Note that $R - V_2RV_2^*$ is the projection of $\mathcal{H}_2$ on $\mathfrak{M} \ominus V_2\mathfrak{M} = \ker S_{\mathfrak{M}}^*$, and $Q = \text{s-lim}_{n\to\infty} V_2^nRV_2^{*n}$ is the projection of $\mathcal{H}_2$ on $\bigcap_1^\infty V_2^n\mathfrak{M}$, which is the unitary subspace of $V_2|\mathfrak{M}$. The result is now easily deduced from Theorem 1.

Corollary. *Let A be (V_1, V_2)-analytic. Let $\mathfrak{M} = (A\mathcal{H}_1)^-$. Then $\mathfrak{M} = B\mathcal{H}_1$, where B is (V_1, V_2)-inner, if and only if $U_{\mathfrak{M}} \prec U_1$.*

Proof. A vector f in $\mathfrak{M}$ belongs to $\mathfrak{M} \ominus V_2\mathfrak{M} = \ker S_{\mathfrak{M}}^*$ if and only if $A^*V_2^*f = 0$, or $V_1^*A^*f = 0$ with $f \in (A\mathcal{H}_1)^-$. Thus $A^*|\ker S_{\mathfrak{M}}^*$ is a one-to-one mapping having range in $\ker S_1^*$. Hence

$$\dim(\ker S_{\mathfrak{M}}^*) \leq \dim(\ker S_1^*)$$

and $S_{\mathfrak{M}} \prec S_1$. The result now follows from the theorem.

4. **Inner-outer factorization.** Let V_1, V_2, V_3 be isometries acting on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ respectively.

Theorem 3. *Assume $S_1 \prec S_2$ and $U_3 \prec U_2$. Then every (V_1, V_3)-analytic operator A has a factorization $A = BC$ where C is (V_1, V_2)-outer and B is (V_2, V_3)-inner with initial space $(C\mathcal{H}_1)^-$. For any factorization $A = BC$ with these properties we have $A^*A = C^*C$ and $C = B^*A$.*

Page [5] obtains a more precise result in a special case.

Proof. Let $\mathfrak{M} = (A\mathcal{H}_1)^-$. The proof of the corollary to Theorem 2 shows that if $V_{\mathfrak{M}} = V_3|\mathfrak{M}$, then $S_{\mathfrak{M}} \prec S_1$. Since $S_1 \prec S_2$ by assumption, we have $S_{\mathfrak{M}} \prec S_2$. Since $U_3 \prec U_2$, $U_{\mathfrak{M}} \prec U_2$. By Theorem 2 there is a (V_2, V_3)-inner operator B such that $(A\mathcal{H}_1)^- = B\mathcal{H}_2$. Then $A = BB^*A$, or $A = BC$ where $C = B^*A$.

We show that the initial space $\mathfrak{N}$ of B reduces V_2. If $f \in \mathfrak{N}$ then $\|BV_2f\| = \|V_3Bf\| = \|Bf\| = \|f\| = \|V_2f\|$, so $V_2f \in \mathfrak{N}$. Hence $V_2\mathfrak{N} \subseteq \mathfrak{N}$. Also $V_2^*\mathfrak{N} = V_2^*B^*\mathcal{H}_3 = B^*V_3^*\mathcal{H}_3 \subseteq B^*\mathcal{H}_3 = \mathfrak{N}$, so the assertion follows.

We show that C is (V_1, V_2)-outer and $(C\mathcal{H}_1)^- = \mathfrak{N}$. Since B^*B is the identity on $\mathfrak{N}$, $B^*BV_2B^* = V_2B^*$. Hence $V_2C = V_2B^*A = B^*BV_2B^*A = B^*V_3BB^*A = B^*V_3A = B^*AV_1 = CV_1$, and so C is (V_1, V_2)-analytic. Clearly $(C\mathcal{H}_1)^- \subseteq B^*\mathcal{H}_3 = \mathfrak{N}$. If $f \in \mathfrak{N}$, $f \perp C\mathcal{H}_1$, then $B^*Bf \perp B^*A\mathcal{H}_1$, and $Bf \perp A\mathcal{H}_1$. But $B\mathcal{H}_2 = (A\mathcal{H}_1)^-$, so $Bf = 0$ and $f = B^*Bf = 0$. Hence $(C\mathcal{H}_1)^- = \mathfrak{N}$ and the existence of a factorization has been established. The last statement in the theorem follows by routine arguments.

5. **Products A^*A.** Let V_1, V_2 be isometries acting on spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let $T \in \mathcal{B}(\mathcal{H}_1)$. A necessary condition for T to have the form $T = A^*A$, where $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is (V_1, V_2)-analytic, is that T be nonnegative and V_1-Toeplitz. To state conditions that are both necessary and sufficient we introduce a

Definition. Let V be an isometry on a Hilbert space $\mathcal{H}$. Let T be a nonnegative V-Toeplitz operator on $\mathcal{H}$. Let $\mathcal{H}_T$ be the closure of the range of $T^{1/2}$, considered as a Hilbert space in the metric of $\mathcal{H}$. Let V_T be the unique isometry on $\mathcal{H}_T$ such that $V_T T^{1/2} f = T^{1/2} V f$ for all $f \in \mathcal{H}$.

Theorem 4. *Let T be a nonnegative V_1-Toeplitz operator on $\mathcal{H}_1$. If $S_1 \prec S_2$, then the following statements are equivalent*:

(i) *$T = A^*A$ for some (V_1, V_2)-analytic operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$,*

(ii) *there is an operator $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $C|(T^{1/2}\mathcal{H}_1)^-$ is one-to-one and $CT^{1/2}$ is (V_1, V_2)-analytic, and*

(iii) *$U_T \prec U_2$.*

In this case we can write $T = C^*C$ where C is (V_1, V_2)-outer.

Condition (iii) generalizes Lowdenslager's condition for the shift case. See [7, p. 192].

Proof. If (i) holds then the polar decomposition of A has the form $A = CT^{1/2}$ where the operator C has the properties listed in (ii).

Let (ii) hold. Define $D \in \mathcal{B}(\mathcal{H}_T, \mathcal{H}_2)$ by $D = C|\mathcal{H}_T$. Then D is one-to-one, and

$$V_2 D T^{1/2} f = V_2 C T^{1/2} f = C T^{1/2} V_1 f = D V_T T^{1/2} f$$

for all $f \in \mathcal{H}_1$. Hence D is (V_T, V_2)-analytic. Let the matrix of D with respect to the decompositions $\mathcal{H}_T = \mathcal{K}_T \oplus \mathcal{L}_T$, $\mathcal{H}_2 = \mathcal{K}_2 \oplus \mathcal{L}_2$ be given by

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.$$

Since $DV_T = V_2 D$ we have $D_{12} U_T = S_2 D_{12}$, and so $D_{12} = 0$. It follows that D_{22} is a one-to-one operator on $\mathcal{L}_T$ to $\mathcal{L}_2$. Also $D_{22} U_T = U_2 D_{22}$. By a result in Douglas [2], the closure of the range of D_{22} reduces U_2, and U_T is unitarily equivalent to the restriction of U_2 to this subspace. Thus $U_T \prec U_2$ and (iii) holds.

Let (iii) hold. We apply Theorem 3 with $\mathcal{H}_3 = \mathcal{H}_T$, $V_3 = V_T$. The operator $Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_T)$ defined by $Yf = T^{1/2} f$, $f \in \mathcal{H}_1$, is (V_1, V_T)-analytic, so by Theorem 3 it has a factorization $Y = BC$ where C is (V_1, V_2)-outer and B is (V_2, V_T)-inner with initial space $(C\mathcal{H}_1)^-$. Also $T = Y^*Y = C^*C$. The theorem follows.

As a consequence we obtain a generalization of a theorem of Page [6].

Corollary. *Assume $S_1 \prec S_2$, and suppose that there exists a (V_1, V_2)-*

analytic operator D with zero kernel. Then every invertible nonnegative V_1-Toeplitz operator T on $\mathcal{H}_1$ has a factorization $T = A^*A$ where A is (V_1, V_2)-outer.

Proof. Condition (ii) in the theorem is satisfied with $C = DT^{-1/2}$.

6. **An extremal problem.** Our last result generalizes Moore's treatment [3] of Szegö's infimum problem.

Theorem 5. *Let V be an isometry on a Hilbert space $\mathcal{H}$. Let T be a nonnegative V-Toeplitz operator on $\mathcal{H}$. Then given $c \in \mathcal{C}$ ($\mathcal{C} = \ker V^*$), we have*

$$\inf_{f \in \mathcal{H}} \langle T(c - Vf), c - Vf \rangle > 0$$

if and only if c has a nonzero projection on $(\mathcal{C} \cap T^{1/2}\mathcal{H})^-$.

Proof. Let $\mathcal{H}_T$ and V_T be defined as in the previous section. It is easy to see that $T^{1/2}\mathcal{C}_T = \mathcal{C} \cap T^{1/2}\mathcal{H}$. Now

$$\langle T(c - Vf), c - Vf \rangle = \|T^{1/2}c - T^{1/2}Vf\|^2 = \|T^{1/2}c - V_T T^{1/2}f\|^2$$

for any $f \in \mathcal{H}$. Since $T^{1/2}\mathcal{H}$ is dense in $\mathcal{H}_T$ and $\mathcal{H}_T = \mathcal{C}_T \oplus V_T\mathcal{H}_T$, the infimum in the theorem is 0 if and only if $T^{1/2}c \perp \mathcal{C}_T$, that is, $c \perp T^{1/2}\mathcal{C}_T$ or $c \perp \mathcal{C} \cap T^{1/2}\mathcal{H}$. The result follows.

REFERENCES

1. A. Devinatz and M. Shinbrot, *General Wiener-Hopf operators*, Trans. Amer. Math. Soc. 145 (1969), 467–494. MR 40 #4800.

2. R. G. Douglas, *On the operator equation $S^*XT = X$ and related topics*, Acta Sci. Math. (Szeged) 30 (1969), 19–32. MR 40 #3347.

3. B. Moore, *The Szegö infimum*, Proc. Amer. Math. Soc. 29 (1971), 55–62. MR 44 #7355a, b; *Erratum*, ibid. 31 (1972), 638.

4. P. S. Muhly, *Inner functions and isometries* (preprint).

5. L. B. Page, *Applications of the Sz.-Nagy and Foiaş lifting theorem,* Indiana Univ. Math. J. 20 (1970/71), 135–145. MR 41 #6003.

6. ———, *Factoring in the commutant of a normal operator* (preprint).

7. M. Rosenblum and J. Rovnyak, *The factorization problem for nonnegative operator functions*, Bull. Amer. Math. Soc. 77 (1971), 287–318. MR 42 #8315.

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