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ON A NONUNIFORM PARABOLIC EQUATION WITH MIXED BOUNDARY CONDITION

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ABSTRACT. This paper discusses the existence of weak solutions for an initial boundary-value problem of a nonuniform second order parabolic equation in which the coefficient $b(t, x)$ of u_t is nonnegative and the coefficient matrix $(a_{ij}(t, x))$ of the elliptic part is not necessarily positive definite. When $b(t, x) \equiv 0$, this problem is reduced to a degenerate elliptic system. A discussion of the existence of weak solutions for the degenerate elliptic boundary-value problem from the parabolic system is included.

1. **Introduction.** Let Ω be a bounded domain in R^n and let $\Gamma \equiv \Gamma_1 \cup \Gamma_2$ be the boundary of Ω. We consider the initial boundary-value problem:

$$Lu \equiv b(t, x)u_t - \sum_{i,j=1}^{n} (a_{ij}(t, x)u_{x_j})_{x_i} + c(t, x)u = f(t, x) \quad (t \in (0, T],\ x \in \Omega), \tag{1.1}$$

$$\partial u/\partial \nu + \beta(t, x)u = 0 \quad (t \in (0, T],\ x \in \Gamma_1),$$
$$u(t, x) = 0 \quad (t \in (0, T],\ x \in \Gamma_2), \tag{1.2}$$

$$u(0, x) = u_0(x) \quad (x \in \Omega), \tag{1.3}$$

where $\beta(t, x) \geq 0$ and $\partial/\partial\nu$ denotes the conormal derivative on Γ_1, that is,

$$\frac{\partial u}{\partial \nu} \equiv \sum_{i,j=1}^{n} n_i(t, x)a_{ij}(t, x)\frac{\partial u}{\partial x_j} \quad (t \in (0, T],\ x \in \Gamma_1),$$

with $(n_1, \cdots, n_n)$ being the outer unit normal vector on Γ_1. It is assumed that $a_{ij} = a_{ji}$ which together with b, b_t, c, f are bounded measurable real

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functions in $D \equiv (0, T] \times \Omega$. The functions β, u_0 are assumed bounded measurable in $(0, T] \times \Gamma_1$ and Ω, respectively. The operator L is uniformly parabolic if the function b is positive and the matrix $A \equiv (a_{ij})$ is positive definite on $\bar{D}$, the closure of D. In this paper, we study a nonuniform parabolic operator in the sense that b is nonnegative and A is positive semidefinite on $\bar{D}$. Specifically, we study the existence of weak solutions for the system (1.1)–(1.3) for the case where

$$b(t, x) \geq 0, \qquad \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j \geq 0, \qquad ((t, x) \in \bar{D},\ \xi = (\xi_1, \cdots, \xi_n) \in R^n). \tag{1.4}$$

Thus either b or a_{ij} (or both) may assure zero values inside the domain D. In addition, we allow either Γ_1 or Γ_2 of the boundary surface Γ to be empty. In this situation, only one of the conditions in (1.2) appears.

When $b(t, x) \equiv 0$ in D the system (1.1)–(1.3) is reduced to the boundary-value problem:

$$-\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_j})_{x_i} + c(x)u = f(x) \qquad (x \in \Omega), \tag{1.5}$$

$$\partial u/\partial \nu + \beta(x)u = 0 \qquad (x \in \Gamma_1), \qquad u(x) = 0 \qquad (x \in \Gamma_2). \tag{1.6}$$

By considering the above nonuniform elliptic system as a degenerate case of the parabolic system (1.1)–(1.3), we deduce a similar result for the problem (1.5), (1.6).

Nonuniform parabolic equations in the form of (1.1)–(1.3) have been studied by Ford [1] for the case where A is a strictly positive scalar function and by Ivanov [2] for the case $b(t, x) \equiv 1$ in D. In both papers, the boundary condition is of Dirichlet type. On the other hand, much work has been done on the degenerate elliptic system (1.5), (1.6). To list a few we refer to the work in [3]–[7]. In most cases, however, the matrix A is assumed to satisfy the condition

$$\nu(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \mu(x)|\xi|^2 \qquad (\xi \in R^n),$$

where ν, μ are positive in Ω and can be zero only on the boundary of Ω. Our assumption allows ν, μ to be zero inside Ω. In fact, our treatment includes the trivial case where A is the zero matrix, that is, $a_{ij}(x) \equiv 0$ in D.

2. **The main results.** Let $H = \{\phi \in C^2(\overline{D});\ \phi(t, x) = 0$ on $(0, T] \times C_2$ and $\phi(T, x) = 0$ in $\overline{\Omega}\}$, where $C^2(\overline{D})$ denotes the set of twice continuously differentiable real functions on $\overline{D}$. For any $\phi, \psi \in H$ we set

$$\langle \phi, \psi \rangle = \int_D \phi(t, x)\psi(t, x)\, dz, \qquad \|\phi\| = \langle \phi, \phi \rangle^{1/2},$$

$$\langle \phi, \psi \rangle_A = \int_D \sum_{i,j=1}^n a_{ij}(t, x)\phi_{x_j}(t, x)\psi_{x_i}(t, x)\, dz, \qquad \|\phi\|_A = \langle \phi, \phi \rangle_A^{1/2},$$

$$\langle \phi, \psi \rangle_\Gamma = \int_0^T \int_{\Gamma_1} \beta(t, x)\phi(t, x)\psi(t, x)\, dz, \qquad \|\phi\|_\Gamma = \langle \phi, \phi \rangle_\Gamma^{1/2},$$

$$\langle \phi, \psi \rangle_b = \int_\Omega b(0, x)\phi(0, x)\psi(0, x)\, dx, \qquad \|\phi\|_b = \langle \phi, \phi \rangle_b^{1/2},$$

where $dz = dtdx$. For $\phi, \psi \in C^2(\overline{\Omega})$, the set of twice differentiable functions which vanish on C_2, we write

$$\langle \phi, \psi \rangle' = \int_\Omega \phi(x)\psi(x)\, dx, \qquad \|\phi\|' = (\langle \phi, \phi \rangle')^{1/2},$$

and similar definitions for $\langle \phi, \psi \rangle_A'$, $\langle \phi, \psi \rangle_\Gamma'$. Define

$$\langle \phi, \psi \rangle_H = \langle \phi, \psi \rangle_A + \langle \phi, \psi \rangle_\Gamma + \langle (c - b_t/2)\phi, \psi \rangle + \tfrac{1}{2}\langle \phi, \psi \rangle_b \qquad (\phi, \psi \in H). \tag{2.1}$$

Then $\langle \cdot, \cdot \rangle_H$ is a symmetric bilinear functional on $H \times H$. Assume, for some constant $\delta > 0$,

$$\langle \phi, \phi \rangle_H \geq \delta \langle \phi, \phi \rangle \qquad (\phi \in H). \tag{2.2}$$

Then $\langle \cdot, \cdot \rangle_H$ defines an inner product in H. We denote the completion of H with respect to the norm $\|\phi\|_H = \langle \phi, \phi \rangle_H^{1/2}$ by H^*.

A function $u \in H^*$ is said to be a weak solution of the problem (1.1)–(1.3) if

$$\langle u, \phi \rangle_A + \langle u, \phi \rangle_\Gamma + \langle u, c\phi \rangle - \langle u, (b\phi)_t \rangle - \langle u_0, \phi \rangle_b = \langle f, \phi \rangle \qquad \text{for all } \phi \in H. \tag{2.3}$$

Our main result for the existence problem of (1.1)–(1.3) is the following:

Theorem 1. *Let $b(t, x) \geq 0$ and the matrix $A \equiv (a_{ij})$ be positive semi-definite in $\overline{D}$. If the condition* (2.2) *holds, then the problem* (1.1)–(1.3) *has a weak solution $u \in H^*$.*

Remarks. (i) The condition (2.2) is fulfilled if there exists a constant $b_0 > 0$ such that

(2.4) $$2c(t, x) - b_t(t, x) \geq b_0 \qquad ((t, x) \in D, \text{ a.e.}).$$

Furthermore, if we transform the problem (1.1)–(1.3) by $u \to e^{-\lambda t}u$, where λ is a constant, then (2.4) may be replaced by the weaker condition: $2(c + \lambda b) - b_t \geq b_0$ a.e. in D for some λ. In particular, if $b(t, x) \geq b_1 > 0$ in D for some constant $b_1 > 0$, then (2.4) (and thus (2.2)) is satisfied by choosing a sufficiently large λ.

(ii) In case the matrix A is positive definite in $\overline{D}$ then

$$\langle \phi, \phi \rangle_A = \int_D \sum_{i,j=1}^{n} a_{ij}\phi_{x_j}\phi_{x_i}\, dz \geq d_0 \int \sum_{i=1}^{n} |\phi_{x_i}|^2\, dz$$

for some constant $d_0 > 0$. Using the inequality

$$\int_D \sum_{i=1}^{n} |\phi_{x_i}|^2\, dz \geq \gamma \int_D |\phi|^2\, dz \qquad (\gamma > 0)$$

for functions ϕ satisfying $\phi(t, x) = 0$ on $(0, T] \times \Gamma$, we obtain $\langle \phi, \phi \rangle_A \geq d_0 \gamma \|\phi\|^2$, where $\gamma > 0$ is a constant depending only on Ω ($\gamma = \pi^2/l^2$ for $\Omega = (0, l)$). Thus (2.2) is satisfied if

(2.5) $$2(c + d_0\gamma) - b_t \geq b_0 \quad \text{in } D \text{ a.e.}$$

In this situation, the problem (1.1)–(1.3) (with $\Gamma = \Gamma_2$) has a weak solution which is a direct extension of the result given in [1]. We remark that since b depends on t, a change of scale in t does not always insure the condition (2.5).

It will be shown in the following section that if we let

(2.6) $$B[u, \phi] = \langle u, \phi \rangle_A + \langle u, \phi \rangle_\Gamma + \langle u, c\phi - (b\phi)_t \rangle \qquad (\phi \in H),$$

then there is a unique closable linear operator $S\colon H \to H^*$ such that

(2.7) $$B[u, \phi] = \langle u, S\phi \rangle_H \quad \text{for all } u \in H^*,\ \phi \in H.$$

Denote the closure of S by $\overline{S}$ and the range of $\overline{S}$ by $R(\overline{S})$; then we have

Theorem 2. *Let the conditions in Theorem* 1 *be satisfied and let* u, v *be any two weak solutions of the problem* (1.1)–(1.3). *Then there is a* $v_0 \in R^{\perp}(\overline{S})$ *such that* $u = v + v_0$, *where* $R^{\perp}(\overline{S}) = \{\psi \in H^*;\ \langle \psi, \phi \rangle = 0$ *for all* $\phi \in R(\overline{S})\}$.

When $b(t, x) \equiv 0$ the last two terms in the bilinear form (2.1) are reduced to $\langle c\phi, \psi \rangle$. This leads to the definition of an inner product on $C^2(\overline{\Omega})$ for the boundary-value problem (1.5), (1.6) by the relation

$$(2.8) \qquad \langle \phi, \psi \rangle'_H = \langle \phi, \psi \rangle'_A + \langle \phi, \psi \rangle'_\Gamma + \langle c\phi, \psi \rangle' \qquad (\phi, \psi \in C^2(\overline{\Omega})).$$

The condition (2.2) is reduced to

$$(2.9) \qquad \langle \phi, \phi \rangle'_H \geq \delta \langle \phi, \phi \rangle' \qquad (\phi \in C^2(\overline{\Omega})),$$

and the equation (2.6) becomes

$$B_1[u, \phi] = \langle u, \phi \rangle'_H \qquad (\phi \in C^2(\overline{\Omega})).$$

We denote the completion of $C^2(\overline{\Omega})$ (open with respect to $\|\phi\|'_H = (\langle \phi, \phi \rangle'_H)^{1/2}$) by $\widetilde{H}$ and say that $u \in \widetilde{H}$ is a weak solution of (1.5), (1.6) if

$$(2.10) \qquad \langle u, \phi \rangle'_A + \langle u, \phi \rangle'_\Gamma + \langle u, c\phi \rangle' = \langle f, \phi \rangle' \qquad (\phi \in C^2(\overline{\Omega})).$$

By considering (2.10) as a degenerate case of (2.3) we obtain

Theorem 3. *Let* $A \equiv (a_{ij})$ *be positive semidefinite on* $\overline{\Omega}$ *and let the condition* (2.9) *be satisfied. Then the problem* (1.5), (1.6) *has a unique weak solution* $u \in \widetilde{H}$.

Remark. The problem (1.5), (1.6) still has a solution even when A is the zero matrix. For instance, if $\Gamma = \Gamma_2$ then the condition (2.9) requires that $c(t, x) \geq c_0 > 0$ in D, and thus the function $u = f/c$ in D and $u = 0$ on $(0, T] \times \Gamma$ is the desired solution.

3. **Proof of the theorems.** Using the definition of $B[u, \phi]$ defined in (2.6), equation (2.3) becomes

$$(3.1) \qquad B[u, \phi] = F_{f,u_0}(\phi) \qquad (\phi \in H),$$

where

$$(3.2) \qquad F_{f,u_0}(\phi) = \langle f, \phi \rangle + \langle u_0, \phi \rangle_b.$$

Thus for the existence problem of (1.1)–(1.3) it suffices to show the existence of $u \in H^*$ satisfying (3.1). For this purpose we prepare the following

Lemma 1. *For each* $\phi \in H$, $B[\cdot, \phi]$ *defines a bounded linear functional on* H. *Furthermore,*

$$(3.3) \qquad B[\phi, \phi] = \|\phi\|_H^2 \qquad (\phi \in H).$$

Proof. Let $\Phi_x = (\phi_{x_1}, \cdots, \phi_{x_n})$, $\Psi_x = (\psi_{x_1}, \cdots, \psi_{x_n})$ and let $(\cdot, \cdot)$ denote the Euclidean inner product in R^n. Since A is symmetric, positive semidefinite there exists a unique symmetric square root $A^{1/2}$ such that $(A\Psi_x, \Phi_x) = (A^{1/2}\Psi_x, A^{1/2}\Phi_x)$. By the Schwarz inequality,

$$
(3.4)\quad \begin{aligned} |\langle \psi, \phi\rangle_A| &= \left|\int_D (A\Psi_x, \Phi_x)\,dz\right| \le \left(\int_D |A^{1/2}\Psi_x|^2\,dz\right)^{1/2}\left(\int_D |A^{1/2}\Phi_x|^2\,dz\right)^{1/2} \\ &= \left(\int_D (A\Psi_x, \Psi_x)\,dz\right)^{1/2}\left(\int_D (A\Phi_x, \Phi_x)\,dz\right)^{1/2} = \|\psi\|_A\|\phi\|_A. \end{aligned}
$$

Since $|\langle \psi, \phi\rangle_\Gamma| \le \|\phi\|_\Gamma\|\psi\|_\Gamma$ and $|\langle \psi, c\phi - (b\phi)_t\rangle| \le \|c\phi - (b\phi)_t\|\,\|\psi\|$ we see from (2.6), (3.4), (2.2) that

$$
(3.5)\qquad |B[\psi, \phi]| \le k_\phi \|\psi\|_H \qquad (\psi \in H),
$$

where k_ϕ is a constant depending only on ϕ and the coefficients of L. Hence $B[\cdot, \phi]$ is a bounded linear functional on H. Equation (3.3) follows from (2.1), (2.6) and the identity

$$
(3.6)\qquad \langle \phi, (b\phi)_t\rangle = \tfrac{1}{2}(\langle b_t\phi, \phi\rangle - \langle \phi, \phi\rangle_b) \qquad (\phi \in H).
$$

This proves the lemma.

Proof of Theorem 1. In view of Lemma 1, we can extend $B[\cdot, \phi]$ to a bounded linear functional on H^*. By the Riesz representation theorem there exists $S\phi \in H^*$ such that

$$
(3.7)\qquad B[u, \phi] = \langle u, S\phi\rangle_H \quad \text{for all } u \in H^*,\ \phi \in H.
$$

Clearly, S is a linear closable operator on H to H^*. Since by Lemma 1 and (3.7),

$$
(3.8)\qquad \langle S\phi, \phi\rangle_H = \langle \phi, \phi\rangle_H \qquad (\phi \in D(S) = H),
$$

we see from the closure property of $\overline{S}$ that

$$
(3.9)\qquad \langle \overline{S}\phi, \phi\rangle_H = \langle \phi, \phi\rangle_H \qquad (\phi \in D(\overline{S})).
$$

This implies that $\overline{S}$ has a continuous inverse and thus, by the closed range theorem, $R(\overline{S}^*) = H^*$, where $\overline{S}^*$ is the adjoint operator on $\overline{S}$. On the other hand, from

$$
|F_{f,u_0}(\phi)| \le \|f\|\,\|\phi\| + \|u_0\|_b\|\phi\|_b \le \gamma\|\phi\|_H \qquad (\phi \in H)
$$

for some $\gamma < \infty$, we can extend F_{f,u_0} to a continuous linear functional on H^*. Hence there exists $v \in H^*$ such that

(3.10) $$F_{f,u_0}(\phi) = \langle v, \phi \rangle_H \quad \text{for all } \phi \in H.$$

Since $R(\bar{S}^*) = H^*$ there exists $u \in D(\bar{S}^*)$ such that $\bar{S}^* u = v$. It follows from (3.7), (3.10) that for any $\phi \in H$,

$$B[u, \phi] = \langle u, \bar{S}\phi \rangle_H = \langle \bar{S}^* u, \phi \rangle_H = \langle v, \phi \rangle_H = F_{f,u_0}(\phi).$$

This shows that u is the desired solution.

Proof of Theorem 2. Since both functions u, v satisfy (3.1) with the same f, u_0 we see from (3.7) that

(3.11) $$0 = B[u - v, \phi] = \langle u - v, S\phi \rangle_H \qquad (\phi \in H).$$

The above relation implies

(3.12) $$\langle u - v, \bar{S}\phi \rangle_H = 0 \qquad (\phi \in D(\bar{S})).$$

Hence $(u - v) \in R^{\perp}(\bar{S})$, which proves the theorem.

Proof of Theorem 3. The proof of existence follows from the same argument as for the problem (1.1)–(1.3) with $b \equiv 0$. The uniqueness problem follows from

$$0 = B_1[u - v, \phi] = \langle u - v, \phi \rangle'_H \qquad (\phi \in C^2(\bar{\Omega}))$$

and the fact that $C^2(\bar{\Omega})$ is dense in $\tilde{H}$.

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