ON A NONUNIFORM PARABOLIC EQUATION WITH MIXED BOUNDARY CONDITION

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ABSTRACT. This paper discusses the existence of weak solutions for an initial boundary-value problem of a nonuniform second order parabolic equation in which the coefficient \( b(t, x) \) of \( u_t \) is nonnegative and the coefficient matrix \( (a_{ij}(t, x)) \) of the elliptic part is not necessarily positive definite. When \( b(t, x) = 0 \), this problem is reduced to a degenerate elliptic system. A discussion of the existence of weak solutions for the degenerate elliptic boundary-value problem from the parabolic system is included.

1. Introduction. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and let \( \Gamma = \Gamma_1 \cup \Gamma_2 \) be the boundary of \( \Omega \). We consider the initial boundary-value problem:

\[
Lu = b(t, x)u_t - \sum_{i,j=1}^{n} (a_{ij}(t, x)u_{x_i}u_{x_j}) + c(t, x)u = f(t, x) \quad (t \in (0, T], x \in \Omega),
\]

\[
\frac{\partial u}{\partial \nu} + \beta(t, x)u = 0 \quad (t \in (0, T], x \in \Gamma_1),
\]

\[
u(t, x) = 0 \quad (t \in (0, T], x \in \Gamma_2),
\]

\[
u(0, x) = u_0(x) \quad (x \in \Omega),
\]

where \( \beta(t, x) \geq 0 \) and \( \partial/\partial \nu \) denotes the conormal derivative on \( \Gamma_1 \), that is,

\[
\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{n} n_i(t, x)a_{ij}(t, x) \frac{\partial u}{\partial x_j} \quad (t \in (0, T], x \in \Gamma_1),
\]

with \( (n_1, \ldots, n_n) \) being the outer unit normal vector on \( \Gamma_1 \). It is assumed that \( a_{ij} = a_{ji} \) which together with \( b, b', c, f \) are bounded measurable real...
functions in $D \equiv (0, T] \times \Omega$. The functions $\beta$, $u_0$ are assumed bounded measurable in $(0, T] \times \Gamma_1$ and $\Omega$, respectively. The operator $L$ is uniformly parabolic if the function $b$ is positive and the matrix $A \equiv (a_{ij})$ is positive definite on $\overline{D}$, the closure of $D$. In this paper, we study a nonuniform parabolic operator in the sense that $b$ is nonnegative and $A$ is positive semidefinite on $\overline{D}$. Specifically, we study the existence of weak solutions for the system (1.1)--(1.3) for the case where

$$b(t, x) \geq 0, \quad \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j \geq 0,$$

(1.4)

$$((t, x) \in \overline{D}, \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n).$$

Thus either $b$ or $a_{ij}$ (or both) may assure zero values inside the domain $D$. In addition, we allow either $\Gamma_1$ or $\Gamma_2$ of the boundary surface $\Gamma$ to be empty. In this situation, only one of the conditions in (1.2) appears.

When $b(t, x) \equiv 0$ in $D$ the system (1.1)--(1.3) is reduced to the boundary-value problem:

$$-\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_j})_{x_i} + c(x)u = f(x) \quad (x \in \Omega),$$

$$\partial u/\partial \nu + \beta(x)u = 0 \quad (x \in \Gamma_1),$$

(1.5)

$$u(x) = 0 \quad (x \in \Gamma_2).$$

(1.6)

By considering the above nonuniform elliptic system as a degenerate case of the parabolic system (1.1)--(1.3), we deduce a similar result for the problem (1.5), (1.6).

Nonuniform parabolic equations in the form of (1.1)--(1.3) have been studied by Ford [1] for the case where $A$ is a strictly positive scalar function and by Ivanov [2] for the case $b(t, x) \equiv 1$ in $D$. In both papers, the boundary condition is of Dirichlet type. On the other hand, much work has been done on the degenerate elliptic system (1.5), (1.6). To list a few we refer to the work in [3]--[7]. In most cases, however, the matrix $A$ is assumed to satisfy the condition

$$\nu(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \mu(x)|\xi|^2 \quad (\xi \in \mathbb{R}^n),$$

where $\nu, \mu$ are positive in $\Omega$ and can be zero only on the boundary of $\Omega$. Our assumption allows $\nu, \mu$ to be zero inside $\Omega$. In fact, our treatment
includes the trivial case where \( A \) is the zero matrix, that is, \( a_{ij}(x) = 0 \) in \( D \).

2. The main results. Let \( H = \{ \phi \in C^2(\overline{D}); \phi(t, x) = 0 \text{ on } (0, T) \times \Omega \text{ and } \phi(T, x) = 0 \text{ in } \Omega \} \), where \( C^2(\overline{D}) \) denotes the set of twice continuously differentiable real functions on \( \overline{D} \). For any \( \phi, \psi \in H \) we set

\[
\langle \phi, \psi \rangle = \int_D \phi(t, x)\psi(t, x) \, dz, \quad \| \phi \| = \langle \phi, \phi \rangle^{1/2},
\]

\[
\langle \phi, \psi \rangle_A = \int_D \sum_{i,j=1}^n a_{ij}(t, x)\phi_x(t, x)\psi_x(t, x) \, dz, \quad \| \phi \|_A = \langle \phi, \phi \rangle_A^{1/2},
\]

\[
\langle \phi, \psi \rangle_\Gamma = \int_0^T \int_\Gamma \beta(t, x)\phi(t, x)\psi(t, x) \, dz, \quad \| \phi \|_\Gamma = \langle \phi, \phi \rangle_\Gamma^{1/2},
\]

\[
\langle \phi, \psi \rangle_b = \int_\Omega b_0(0, x)\phi(0, x)\psi(0, x) \, dx, \quad \| \phi \|_b = \langle \phi, \phi \rangle_b^{1/2},
\]

where \( dz = dt dx \). For \( \phi, \psi \in C^2(\overline{\Omega}) \), the set of twice differentiable functions which vanish on \( C_2 \), we write

\[
\langle \phi, \psi \rangle' = \int_\Omega \phi(x)\psi'(x) \, dx, \quad \| \phi \|' = ((\phi, \phi)' )^{1/2},
\]

and similar definitions for \( \langle \phi, \psi \rangle_A', \langle \phi, \psi \rangle_\Gamma' \). Define

\[
(2.1) \quad \langle \phi, \psi \rangle_H = \langle \phi, \psi \rangle_A + \langle \phi, \psi \rangle_\Gamma + \langle (c - b_1/2)\phi, \psi \rangle + \frac{1}{2} \langle \phi, \psi \rangle_b
\]

Then \( \langle \cdot, \cdot \rangle_H \) is a symmetric bilinear functional on \( H \times H \). Assume, for some constant \( \delta > 0 \),

\[
(2.2) \quad \langle \phi, \phi \rangle_H \geq \delta \langle \phi, \phi \rangle \quad (\phi \in H).
\]

Then \( \langle \cdot, \cdot \rangle_H \) defines an inner product in \( H \). We denote the completion of \( H \) with respect to the norm \( \| \phi \|_H = \langle \phi, \phi \rangle_H^{1/2} \) by \( H^* \).

A function \( u \in H^* \) is said to be a weak solution of the problem (1.1)—(1.3) if

\[
(2.3) \quad \langle u, \phi \rangle_A + \langle u, \phi \rangle_\Gamma + \langle u, c\phi \rangle - \langle u, (b\phi)_t \rangle - \langle u_0, \phi \rangle_b = \langle f, \phi \rangle \quad \text{for all } \phi \in H.
\]

Our main result for the existence problem of (1.1)—(1.3) is the following:

**Theorem 1.** Let \( b(t, x) \geq 0 \) and the matrix \( A = (a_{ij}) \) be positive semi-definite in \( \overline{D} \). If the condition (2.2) holds, then the problem (1.1)—(1.3) has a weak solution \( u \in H^* \).
Remarks. (i) The condition (2.2) is fulfilled if there exists a constant 
\[ b_0 > 0 \] 
such that
\[ 2c(t, x) - b_t(t, x) \geq b_0 \quad ((t, x) \in D, \text{a.e.}). \] 
Furthermore, if we transform the problem (1.1)–(1.3) by \( u \rightarrow e^{-\lambda t} u \), where \( \lambda \) is a constant, then (2.4) may be replaced by the weaker condition:
\[ 2(c + \lambda b) - b_t \geq b_0 \quad \text{a.e. in } D \text{ for some } \lambda. \] 
In particular, if \( b(t, x) \geq b_1 > 0 \) in \( D \) for some constant \( b_1 > 0 \), then (2.4) (and thus (2.2)) is satisfied by choosing a sufficiently large \( \lambda \).

(ii) In case the matrix \( A \) is positive definite in \( \overline{D} \) then
\[ \langle \phi, \phi \rangle_A = \int_D \sum_{i,j=1}^n a_{ij} \phi_{x_i} \phi_{x_j} \, dz \geq d_0 \int \sum_{i=1}^n |\phi_{x_i}|^2 \, dz \]
for some constant \( d_0 > 0 \). Using the inequality
\[ \int_D \sum_{i=1}^n |\phi_{x_i}|^2 \, dz \geq \gamma \int_D |\phi|^2 \, dz \quad (\gamma > 0) \]
for functions \( \phi \) satisfying \( \phi(t, x) = 0 \) on \((0, T] \times \Gamma\), we obtain \( \langle \phi, \phi \rangle_A \geq d_0 \gamma \|\phi\|^2 \), where \( \gamma > 0 \) is a constant depending only on \( \Omega \) (\( \gamma = \pi^2/l^2 \) for \( \Omega = (0, l)^n \)). Thus (2.2) is satisfied if
\[ 2(c + d_0 \gamma) - b_t \geq b_0 \quad \text{in } D \text{ a.e.} \]
In this situation, the problem (1.1)–(1.3) (with \( \Gamma = \Gamma_2 \)) has a weak solution which is a direct extension of the result given in [1]. We remark that since \( b \) depends on \( t \), a change of scale in \( t \) does not always insure the condition (2.5).

It will be shown in the following section that if we let
\[ B[u, \phi] = \langle u, \phi \rangle_A + \langle u, \phi \rangle_T + \langle u, c \phi - (b \phi)_t \rangle \quad (\phi \in H), \]
then there is a unique closable linear operator \( S: H \to H^* \) such that
\[ B[u, \phi] = \langle u, S \phi \rangle_H \quad \text{for all } u \in H^*, \phi \in H. \]
Denote the closure of \( S \) by \( \overline{S} \) and the range of \( \overline{S} \) by \( R(\overline{S}) \); then we have

Theorem 2. Let the conditions in Theorem 1 be satisfied and let \( u, v \) be any two weak solutions of the problem (1.1)–(1.3). Then there is a \( v_0 \in R(\overline{S})^* \) such that \( u = v + v_0 \), where \( R(\overline{S})^* \) is the dual space of \( R(\overline{S}) \).
When \( b(t, x) = 0 \) the last two terms in the bilinear form (2.1) are reduced to \( \langle c\phi, \psi \rangle \). This leads to the definition of an inner product on \( C^2(\Omega) \) for the boundary-value problem (1.5), (1.6) by the relation

\[
(2.8) \quad \langle \phi, \psi \rangle_H = \langle \phi, \psi \rangle_A + \langle \phi, \psi \rangle_T + \langle c\phi, \psi \rangle \quad (\phi, \psi \in C^2(\Omega)).
\]

The condition (2.2) is reduced to

\[
(2.9) \quad \langle \phi, \phi \rangle_H \geq \delta \langle \phi, \phi \rangle \quad (\phi \in C^2(\Omega)),
\]

and the equation (2.6) becomes

\[
B_1[u, \phi] = \langle u, \phi \rangle_H \quad (\phi \in C^2(\Omega)).
\]

We denote the completion of \( C^2(\Omega) \) (open with respect to \( \|\phi\|_H = \langle \langle \phi, \phi \rangle \rangle^{1/2} \)) by \( \tilde{H} \) and say that \( u \in \tilde{H} \) is a weak solution of (1.5), (1.6) if

\[
(2.10) \quad \langle u, \phi \rangle_A' + \langle u, \phi \rangle_T' + \langle u, c\phi \rangle' = \langle f, \phi \rangle' \quad (\phi \in C^2(\Omega)).
\]

By considering (2.10) as a degenerate case of (2.3) we obtain

**Theorem 3.** Let \( A = (a_{ij}) \) be positive semidefinite on \( \Omega \) and let the condition (2.9) be satisfied. Then the problem (1.5), (1.6) has a unique weak solution \( u \in H \).

**Remark.** The problem (1.5), (1.6) still has a solution even when \( A \) is the zero matrix. For instance, if \( \Gamma = \Gamma_2 \) then the condition (2.9) requires that \( c(t, x) \geq c_0 > 0 \) in \( D \), and thus the function \( u = f/c \) in \( D \) and \( u = 0 \) on \( (0, T) \times \Gamma \) is the desired solution.

3. **Proof of the theorems.** Using the definition of \( B[u, \phi] \) defined in (2.6), equation (2.3) becomes

\[
(3.1) \quad B[u, \phi] = F_{f, u_0}(\phi) \quad (\phi \in H),
\]

where

\[
(3.2) \quad F_{f, u_0}(\phi) = \langle f, \phi \rangle + \langle u_0, \phi \rangle_b.
\]

Thus for the existence problem of (1.1)—(1.3) it suffices to show the existence of \( u \in H^* \) satisfying (3.1). For this purpose we prepare the following

**Lemma 1.** For each \( \phi \in H, B[\cdot, \phi] \) defines a bounded linear functional on \( H \). Furthermore,

\[
(3.3) \quad B[\phi, \phi] = \|\phi\|_H^2 \quad (\phi \in H).
\]
Proof. Let \( \Phi_x = (\phi_{x_1}, \ldots, \phi_{x_n}) \), \( \Psi_x = (\psi_{x_1}, \ldots, \psi_{x_n}) \) and let \((\cdot, \cdot)\) denote the Euclidean inner product in \( \mathbb{R}^n \). Since \( A \) is symmetric, positive semidefinite there exists a unique symmetric square root \( A^{\frac{1}{2}} \) such that 

\[
(A\Phi_x, \Phi_x) = (A^{\frac{1}{2}} \Psi_x, A^{\frac{1}{2}} \Phi_x).
\]

By the Schwarz inequality,

\[
(\psi, \phi)_A = \left| \int_D (A\Psi_x, \Phi_x) \, dz \right| \leq \left( \int_D |A^{\frac{1}{2}} \Psi_x|^2 \, dz \right)^{\frac{1}{2}} \left( \int_D |A^{\frac{1}{2}} \Phi_x|^2 \, dz \right)^{\frac{1}{2}}
= \left( \int_D (A\Psi_x, \Psi_x) \, dz \right)^{\frac{1}{2}} \left( \int_D (A\Phi_x, \Phi_x) \, dz \right)^{\frac{1}{2}} = \|\psi\|_A \|\phi\|_A.
\]

Since \(|(0, 0)| < \|0\|_r \|0\|_r\) and \(|(0, c_0 - (0, c_0)| < \|c_0 - (c_0)| \|0\|_r\|0\|_r\|
we see from (2.6), (3.4), (2.2) that

\[
|\langle \psi, \phi \rangle| \leq k_\phi \|\psi\|_H \quad (\psi \in H),
\]

where \( k_\phi \) is a constant depending only on \( \phi \) and the coefficients of \( L \).

Hence \( B[\cdot, \phi] \) is a bounded linear functional on \( H \). Equation (3.3) follows from (2.1), (2.6) and the identity

\[
\langle \phi, (b\phi)_t \rangle = \frac{1}{2} \langle \langle b_t \phi, \phi \rangle - \langle \phi, \phi \rangle \rangle \quad (\phi \in H).
\]

This proves the lemma.

Proof of Theorem 1. In view of Lemma 1, we can extend \( B[\cdot, \phi] \) to a bounded linear functional on \( H^* \). By the Riesz representation theorem there exists \( S\phi \in H^* \) such that

\[
B[u, \phi] = \langle u, S\phi \rangle_H \quad \text{for all } u \in H^*, \phi \in H.
\]

Clearly, \( S \) is a linear closable operator on \( H \) to \( H^* \). Since by Lemma 1 and (3.7),

\[
\langle S\phi, \phi \rangle_H = \langle \phi, \phi \rangle_H \quad (\phi \in D(S) = H),
\]

we see from the closure property of \( \overline{S} \) that

\[
\langle \overline{S}\phi, \phi \rangle_H = \langle \phi, \phi \rangle_H \quad (\phi \in D(\overline{S})).
\]

This implies that \( \overline{S} \) has a continuous inverse and thus, by the closed range theorem, \( R(\overline{S}^*) = H^* \), where \( \overline{S}^* \) is the adjoint operator on \( \overline{S} \). On the other hand, from

\[
|F_{f, u_0}(\phi)| \leq \|f\| \|\phi\| + \|u_0\|_b \|\phi\|_b \leq \gamma \|\phi\|_H \quad (\phi \in H)
\]

for some \( \gamma < \infty \), we can extend \( F_{f, u_0} \) to a continuous linear functional on \( H^* \). Hence there exists \( \psi \in H^* \) such that
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(3.10) \[ F_{f,u_0}(\phi) = \langle v, \phi \rangle_H \] for all \( \phi \in H \).

Since \( R(\bar{S}^*) = H^* \) there exists \( u \in D(\bar{S}^*) \) such that \( \bar{S}^* u = v \). It follows from (3.7), (3.10) that for any \( \phi \in H \),

\[ B[u, \phi] = \langle u, \bar{S}\phi \rangle_H = \langle \bar{S}^* u, \phi \rangle_H = \langle v, \phi \rangle_H = F_{f,u_0}(\phi). \]

This shows that \( u \) is the desired solution.

**Proof of Theorem 2.** Since both functions \( u, v \) satisfy (3.1) with the same \( f, u_0 \) we see from (3.7) that

(3.11) \[ 0 = B[u - v, \phi] = \langle u - v, S\phi \rangle_H \quad (\phi \in H). \]

The above relation implies

(3.12) \[ \langle u - v, \bar{S}\phi \rangle_H = 0 \quad (\phi \in D(\bar{S})). \]

Hence \( u - v \in R(\bar{S}) \), which proves the theorem.

**Proof of Theorem 3.** The proof of existence follows from the same argument as for the problem (1.1)–(1.3) with \( b \equiv 0 \). The uniqueness problem follows from

\[ 0 = B_1[u - v, \phi] = \langle u - v, \phi \rangle_H \quad (\phi \in C^2(\Omega)) \]

and the fact that \( C^2(\Omega) \) is dense in \( \tilde{H} \).

REFERENCES