LINEAR CONNECTIONS AND ALMOST COMPLEX STRUCTURES
JEAN-MARC TERRIER

ABSTRACT. An almost complex structure is defined on $P$, the principal bundle of linear frames over an arbitrary even-dimensional smooth manifold $M$ with a given linear connection. Complexifying connections are those which induce a complex structure on $P$. For two-dimensional manifolds, every linear connection is of this kind.

In the special case where $M$ itself is an almost complex manifold, a relationship between the two almost complex structures is found and provides a very simple proof of the fact that the existence of an almost complex connection without torsion implies the integrability of the given almost complex structure. As a second application, we give a geometrical interpretation of an identity between the torsion of an almost complex structure on $M$ and the torsion of an almost complex connection over $M$.

1. Introduction. In this paper we associate to each linear connection over an arbitrary even-dimensional smooth manifold $M$ an almost complex structure $\hat{J}$ on $P$, the principal bundle of linear frames over $M$. This almost complex structure actually depends on three objects: a linear connection, a complex structure on $\mathbb{R}^m$ ($m = 2n = \dim M$), and a complex structure on the Lie algebra of the general linear group $GL(m; \mathbb{R})$. Theorem 2.2 gives the explicit relationship. Theorem 3.2 expresses the torsion of the almost complex structure $\hat{J}$ in terms of the curvature and the torsion of the connection via a construction—the mixed torsion map—which turns out to be, by Theorem 3.3, a singular endomorphism of 2-forms on $P$ with values in a complex vector space. This suggests calling complexifying connections those connections which induce a complex structure on $P$. For instance, Theorem 3.4 shows that any linear connection on a two-dimensional manifold is a complexifying connection.
For the rest of the paper, we shall assume that $M$ admits an almost complex structure $J_M$. Theorem 4.1 gives a way to compare $J_M$ with $\hat{J}$. In the case of an almost complex connection, formula (3.1) first provides a direct proof of the converse of Proposition 1.2 in [3] or of the necessary part of Corollary 3.5 in [2]. Secondly, it gives a geometric interpretation of the fact that the torsion $T$ of an almost complex connection on an almost complex manifold $(M, J_M)$ satisfies the identity [2, Proposition 3.6]

$$(1.1) \quad T^M(X, Y) = T(J_MX, J_MY) - T(X, Y) - J_M T(J_M X, Y) - J_M T(X, J_M Y).$$

The reader is referred to [1, Chapters I–IV] and [2, Chapter IX] for basic notions and notations.

2. Almost complex structures on the bundle of linear frames. Let us consider an even-dimensional smooth manifold, $\dim M = m = 2n$. Let $P$ be the principal bundle of linear frames over $M$ with projection $\pi$. Denote by $J$ (resp. $J_0$) the canonical complex structure of $\mathbb{R}^m$ [2, p. 115] (resp. $\mathfrak{gl}(m; \mathbb{R})$, the Lie algebra of the general linear group $\text{GL}(m; \mathbb{R})$, identified with the Lie algebra of all $m \times m$ real matrices). $J$ is represented in the standard basis of $\mathbb{R}^m$ by the matrix

$$
\begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix},
$$

where $I_n$ is the identity $n \times n$ matrix; $J_0$ acts on $\mathfrak{gl}(m; \mathbb{R})$ by left multiplication with the same matrix.

Suppose now we are given a linear connection $\Gamma$ in $P$ with $\omega$ as its 1-form of connection. Using the fact that $\Gamma$ induces in each point $u \in P$ a direct sum decomposition of $T_u(P)$, the tangent space of $P$ at $u$, into a vertical and a horizontal part, we make the following definition.

**Definition 2.1.** Let us call $\hat{J}$ the $(1, 1)$-tensor field on $P$ given by

$$(2.1) \quad \hat{J}X = \lambda J_0 \omega X + BJ\theta X \quad \forall u \in P, \forall X \in T_u(P),$$

where

(i) $\theta$ is the canonical $\mathbb{R}^m$-valued 1-form on $P$,

(ii) $\lambda$ is the isomorphism between $\mathfrak{gl}(m; \mathbb{R})$ and the fundamental vector fields on $P$,

(iii) $B(\xi)$ is the standard horizontal vector field on $P$ corresponding to $\xi \in \mathbb{R}^m$, whose value at $u$ is the unique horizontal vector such that
\[ \pi(B(\xi_u)) = u(\xi), \] where \( u \) is interpreted as a linear isomorphism: \( \mathbb{R}^m \to T\pi(u)(M) \).

We have the following theorem.

**Theorem 2.2.** If \( M \) is an even-dimensional smooth manifold with linear connection \( \Gamma \), \( J \) (resp. \( J_0 \)) the canonical complex structure on \( \mathbb{R}^m \) (resp. on \( \mathfrak{gl}(m; \mathbb{R}) \)), then:

1. the \((1, 1)\)-tensor field \( \hat{\gamma} \) defined on \( P \) by
   \[ \hat{\gamma} = \lambda J_0 \omega + BJ\theta \]
is an almost complex structure on \( P \).

2. If \( \omega' = \omega + \eta \) is another connection 1-form with \( \eta \) a tensorial form of type \( \text{adj GL}(m; \mathbb{R}) \), then the corresponding almost complex structure \( \hat{\gamma}' \) on \( P \) is given by
   \[ \hat{\gamma}' = \hat{\gamma} + \lambda (J_0 \eta - \eta BJ\theta). \]

3. If \( J_S \) (resp. \( J_Q \)) is a complex structure of \( \mathbb{R}^m \) (resp. \( \mathfrak{gl}(m; \mathbb{R}) \)) corresponding to the nontrivial left coset \([S]\) in \( \text{GL}(m; \mathbb{R})/\text{GL}(n; \mathbb{C}) \) (resp. \([Q]\) in \( \text{GL}(m^2; \mathbb{R})/\text{GL}(2m^2; \mathbb{C}) \)), then the induced almost complex structure on \( P \) is given by
   \[ \hat{\gamma}_{Q,S} = -(\lambda J_Q J_0 \omega + BJ_S J_0 \theta). \]

**Proof.** (1) Because \( \omega \) (resp. \( \theta \)) is a vertical (resp. horizontal) form and \( \theta B = 1 \), we easily have \( \hat{\gamma}^2 = -1 \).

(2) A standard horizontal vector field \( B' \) in the new connection \( \omega' = \omega + \eta \) is given by \( B' = B - \lambda \eta B \), because, for each \( \xi \in \mathbb{R}^m \), \( \eta B'(\xi) = -\omega B'(\xi) \) and \( \omega \lambda + B\theta = 1 \).

(3) By definition, \( \hat{\gamma}_{Q,S} = \lambda J_Q \omega + BJ_S \theta \), where \( J_Q = QJS^{-1} \) and \( J_S = JSJ^{-1} \). The formula follows because \( \omega \hat{\gamma} = J_0 \omega \) and \( \theta \hat{\gamma} = J_0 \theta \).

From now on, we shall refer to the almost complex structure \( \hat{\gamma} \) defined above as the canonical almost complex structure corresponding to the condition \( \omega \) or simply the \( \omega \)-canonical almost complex structure on \( P \).

3. **Complexifying connections.** It is known that a necessary and sufficient condition for an almost complex structure \( \hat{\gamma} \) to be integrable is the vanishing of its torsion \( \hat{\tau} \) defined by

\[ \hat{\tau}(X, Y) = [X, Y] + \hat{\gamma}([X, Y] + \hat{\gamma}(X, Y) - [\hat{\gamma}(X, Y), [X, Y]]. \]

where \( X \) and \( Y \) are vector fields on \( P \).

**Definition 3.1.** For each \( \alpha \in \Lambda^2(P, \mathcal{V}) \), a 2-form on \( P \) with values in
a vector space $V$ with a complex structure $J_V$, let $\tilde{\alpha} \in \Lambda^2(P, V)$ defined by

$$-\frac{1}{2} \tilde{\alpha}(X, Y) = \alpha(X, Y) + J_V \alpha(JX, Y) + J_V \alpha(X, JY) - \alpha(JX, JY).$$

**Theorem 3.2.** If $\Gamma$ is a linear connection over $M$ with $\omega$ as its 1-form of connection, then the torsion $\tilde{\tau}$ of the $\omega$-canonical almost complex structure $\tilde{\tau}$ on $P$ is given by

$$\tilde{\tau} = \lambda \tilde{\Omega} + B \tilde{\Theta},$$

where $\Omega$ (resp. $\Theta$) is the curvature form (resp. the torsion form of $\Gamma$).

**Proof.** Since each connection induces a parallelization of $P$ by means of the vector fields $B_i = B(e_i)$, $1 \leq i \leq m$, where $\{e_1, \cdots, e_m\}$ is the standard basis of $\mathbb{R}^m$, and the vector fields $X_i^j = \lambda E_i^j$, $1 \leq i, j \leq m$, where $(E_i^j)_{1 \leq i, j \leq m}$ is the basis of $\mathfrak{gl}(m; \mathbb{R})$ defined by $(E_i^j)_{\gamma} = \delta_i^\gamma \delta_j^\gamma$, it is sufficient to compute the torsion in the three following cases:

1. $X$ and $Y$ are horizontal; more precisely $X = B_1$ and $Y = B_2$, where $B_i = B(\xi_i)$, $\xi_1$ and $\xi_2$ linearly independent in $\mathbb{R}^m$.
2. $X$ is horizontal; $X = B(\xi)$ for $\xi$ nonzero in $\mathbb{R}^m$ and $Y$ is vertical, say $Y = \lambda A$ where $A$ nonzero in $\mathfrak{gl}(m, \mathbb{R})$.
3. $X$ and $Y$ are vertical; $X = \lambda A_1$ and $Y = \lambda A_2$ with $A_1$ and $A_2$ linearly independent in $\mathfrak{gl}(m, \mathbb{R})$.

Using the structure equations [1, p. 120] for $\Omega$ and $\Theta$, we first get

$$[B_1, B_2] = -2\lambda \Omega(B_1, B_2) - 2B \Theta(B_1, B_2).$$

Secondly by [1, p. 119] we have

$$[B(\xi), \lambda A] = -B(A(\xi)),$$

and finally,

$$[\lambda A_1, \lambda A_2] = \lambda[A_1, A_2].$$

Observing that $\tilde{J}B(\xi) = B(J\xi)$ since $\Theta B = 1$, and $\tilde{J}\lambda A = \lambda J_0 A$ since $\omega \lambda = 1$, the theorem follows by computation and use of Definition 2.1.

This result suggests calling a complexifying connection a connection whose curvature and torsion satisfy the conditions $\tilde{\Omega} = 0$ and $\tilde{\Theta} = 0$. Then Theorem 3.2 can be stated in the following way:

**Theorem 3.2'.** The $\omega$-canonical almost complex structure on $P$ induced by a connection $\Gamma$ is integrable if and only if $\Gamma$ is a complexifying connection.

A flat linear connection on $M$, without torsion, gives a (rather trivial) example of such a connection, since $\Omega = 0$ (resp. $\Theta = 0$) implies $\tilde{\Omega} = 0$ (resp. $\tilde{\Theta} = 0$). However, it is important to note that, in general, the form $\tilde{\alpha}$ can be zero without $\alpha$ being zero. More precisely we have the following theorem.
Let us first call mixed torsion map the endomorphism $\mu$ of $\Lambda^2 (P, V)$ defined by $\mu a = -\frac{1}{2} \bar{a}$. The name is suggested by the formal resemblance to Definition 3.1 of $\tilde{\tau}$, the structures $J_V$ and $\hat{J}$ being "mixed" together, and also by formula (3.2).

**Theorem 3.3.** The mixed torsion map $\mu$ is singular.

**Proof.** The fact that $\mu$ is linear is trivial. To show that the kernel of $\mu$ is different from zero, let us first note that $\mu^2 - 4\mu = 0$ by an easy computation. On the other hand, $\mu \neq 4I$ by the Lemma below (whose proof was kindly suggested to the author by Professor S. Takahashi). Suppose now that $\mu$ is nonsingular, then $(\mu - 4)a$ is in the kernel of $\mu$ for each $a$, hence so is $\mu a = 4a$, which gives a contradiction. Q.E.D.

**Lemma.** For the mixed torsion map $\mu$ we have $\mu \neq 4I$.

**Proof.** Write $V = V_0 \oplus J_V V_0$, viewing $V$ as a complex space and $V_0$ as the set of $a \in V$ such that $\bar{a} = a$ (bar denotes complex conjugation). Accordingly we write $a = a^+ + J_V a^-$ for each $a \in V$. It is sufficient to show that $\mu = 4I$ and $a^- = 0$ implies $a^+ = 0$. By definition of $\mu$, we have

$$3a(X, Y) = J_V a(\hat{J}X, Y) + J_V a(X, \hat{J}Y) - a(\hat{J}X, \hat{J}Y).$$

Thus,

$$3a^+(X, Y) = -a^+(\hat{J}X, \hat{J}Y) \quad \text{and} \quad a^+(X, Y) = a^+(\hat{J}X, \hat{J}Y).$$

Hence $4a^+(X, Y) = 0$. Q.E.D.

For low-dimensional manifolds we have the following result.

**Theorem 3.4.** Each linear connection on a 2-dimensional manifold is a complexifying connection.

**Proof.** Take any nonzero element in $\mathbb{R}^2$ and call $B$ the standard horizontal vector field corresponding to it. By Theorem 3.2, it is sufficient to show that $\bar{\alpha}(B, \hat{J}B) = 0$, where $\alpha = \Omega$ (resp. $\Theta$), because these are horizontal 2-forms. By the skew symmetry of $\alpha$ and Definition 3.1 we have

$$-\frac{1}{2} \bar{\alpha}(B, \hat{J}B) = \alpha(B, \hat{J}B) + J_V \alpha(\hat{J}B, \hat{J}B) - J_V \alpha(B, B) + \alpha(\hat{J}B, B) = 0.$$ Q.E.D.

**Remark.** Using the parallelization of $P$ given by a connection $\Gamma$, we can introduce a Riemannian metric on $P$ in a standard way, and it is easy to verify that with respect to this metric, $\Gamma$ induces an almost hermitian structure on $P$, which is hermitian in case $\Gamma$ is a complexifying connection. How-
ever, such an (almost) hermitian structure is never (almost) Kählerian, as one can easily check.

4. Almost complex structures, up and down. From now on, we shall assume that $M$, itself, admits an almost complex structure $J_M$. It is known that the existence of an almost complex structure $J_M$ on $M$ is equivalent to the reduction of $L(M)$ to $C(M)$, the bundle of complex linear frames, the subbundle of $L(M)$ defined by $C(M) = \{ u \in L(M); uJ = f_M u \}$. (We view $u$ as in Definition 2.1(iii).) Let $\omega$ be any linear connection on $M$. If $X$ is a vector field on $X$ and $X^*$ its horizontal lift to $L(M)$, we can compare $\hat{J}X^*$ with $(J_M X)^*$ in the following theorem.

**Theorem 4.1.** For each vector field $X$ on $M$, we have $\hat{J}X^* = (J_M X)^*$ on $C(M)$.

**Proof.** For an arbitrary point $u$ in $L(M)$ we have by definition of $J$, $$(\hat{J}X^*)_u = (BJ(\theta(X^*))_u = Y,$$ the unique horizontal vector at $u$ such that $\pi Y = uJ(\theta(X_u) = uJ u^{-1}X_{\pi(u)}$. On the other side $(J_M X)_u^* = Z$, the unique horizontal vector at $u$ such that $\pi Z = J_M X_{\pi(u)}$. Therefore

$$Y = Z \iff uJ u^{-1}X_{\pi(u)} = J_M X_{\pi(u)} \iff uJ u^{-1} = J_M \iff u \in C(M). \quad Q.E.D.$$ 

From now on we shall assume that $\omega$ is the 1-form of an almost complex connection, i.e. a linear connection arising from a connection on $C(M)$. This is known to be equivalent with the fact that $J_M$ is parallel with respect to this connection. Still denoting by $\hat{J}$ the corresponding almost complex structure on $L(M)$, we have the following relationship between its torsion $\hat{r}$ and $r_M$, the torsion of $J_M$.

**Theorem 4.2.** If $(M, J_M)$ is an almost complex manifold, and if $\omega$ is the 1-form of an almost complex connection on $M$, then

$$B\Theta(X^*, Y^*) = (r_M(X, Y))^*$$

on $C(M)$, where $X$ and $Y$ are vector fields on $M$.

**Proof.** According to (3.2) the torsion of $\hat{J}$ is given by $\hat{r}(X^*, Y^*) = \hat{r}(X^*, Y^*) + B\Theta(X^*, Y^*)$. We are interested only in the second term of the right member, the first one being in the kernel of the differential of $\pi$. Formula (3.1) gives

$$\hat{r}(X^*, Y^*) = [X^*, Y^*] + \hat{J}[\hat{J}X^*, Y^*] + \hat{J}[X^*, \hat{J}Y^*] - [\hat{J}X^*, \hat{J}Y^*].$$

If we restrict ourselves to $C(M)$ then, by Theorem 2.2 and the very defini-
tion of an almost complex connection, the horizontal part of the right member of this equation becomes

\[ [X, Y]^* + (J_M[X, Y])^* + (J_M[X, J_M Y])^* - [J_M X, J_M Y]^*, \]

which is exactly the horizontal lift of the vector field \( \tau_M(X, Y) \). Q.E.D.

This theorem has two interesting and simple applications. The first one is the converse (in the case of almost complex structures) of the general theorem [3], that if a \( G \)-structure on \( M \) is integrable it admits a torsionfree connection. We have

**Corollary 4.3.** If an almost complex manifold \( (M, J_M) \) admits a torsionfree almost complex connection, then \( J_M \) is integrable.

**Proof.** \( \Theta = 0 \) implies \( \tilde{\Theta} = 0 \) and, by Theorem 4.2, \( \tau_M = 0 \).

The second application is a geometric interpretation of the identity (1.1) which is simply the projection down to \( M \) of the torsion of \( \tilde{\jmath} \) evaluated on horizontal lifts in \( C(M) \). This is a consequence of the definition of the torsion \( T \) of the connection as \( T(X, Y) = u2\Theta(X^*, Y^*) \) and of \( \tilde{\Theta} \) as

\[ -\frac{1}{2} \tilde{\Theta}(X^*, Y^*) = \Theta(X^*, Y^*) + J\Theta(\tilde{\jmath} X^*, Y^*) + J\Theta(X^*, \tilde{\jmath} Y^*) - \Theta(\tilde{\jmath} X^*, \tilde{\jmath} Y^*). \]

**REFERENCES**


DéPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE MONTÉRÉAL, MONTÉRÉAL, QUÉBEC, CANADA