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LINEAR CONNECTIONS AND ALMOST COMPLEX STRUCTURES

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ABSTRACT. An almost complex structure is defined on P, the principal bundle of linear frames over an arbitrary even-dimensional smooth manifold M with a given linear connection. *Complexifying connections* are those which induce a complex structure on P. For two-dimensional manifolds, every linear connection is of this kind.

In the special case where M itself is an almost complex manifold, a relationship between the two almost complex structures is found and provides a very simple proof of the fact that the existence of an almost complex connection without torsion implies the integrability of the given almost complex structure. As a second application, we give a geometrical interpretation of an identity between the torsion of an almost complex structure on M and the torsion of an almost complex connection over M.

1. **Introduction.** In this paper we associate to each linear connection over an arbitrary even-dimensional smooth manifold M an almost complex structure $\hat{J}$ on P, the principal bundle of linear frames over M. This almost complex structure actually depends on three objects: a linear connection, a complex structure on $\mathbf{R}^m$ ($m = 2n = \dim M$), and a complex structure on the Lie algebra of the general linear group $GL(m; \mathbf{R})$. Theorem 2.2 gives the explicit relationship. Theorem 3.2 expresses the torsion of the almost complex structure $\hat{J}$ in terms of the curvature and the torsion of the connection via a construction—the mixed torsion map—which turns out to be, by Theorem 3.3, a singular endomorphism of 2-forms on P with values in a complex vector space. This suggests calling *complexifying connections* those connections which induce a complex structure on P. For instance, Theorem 3.4 shows that any linear connection on a two-dimensional manifold is a complexifying connection.

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For the rest of the paper, we shall assume that M admits an almost complex structure J_M. Theorem 4.1 gives a way to compare J_M with $\hat{J}$. In the case of an almost complex connection, formula (3.1) first provides a direct proof of the converse of Proposition 1.2 in [3] or of the necessary part of Corollary 3.5 in [2]. Secondly, it gives a geometric interpretation of the fact that the torsion T of an almost complex connection on an almost complex manifold (M, J_M) satisfies the identity [2, Proposition 3.6]

$$(1.1)\quad \tau_M(X, Y) = T(J_M X, J_M Y) - T(X, Y) - J_M T(J_M X, Y) - J_M T(X, J_M Y).$$

The reader is referred to [1, Chapters I–IV] and [2, Chapter IX] for basic notions and notations.

2. Almost complex structures on the bundle of linear frames. Let us consider an even-dimensional smooth manifold, $\dim M = m = 2n$. Let P be the principal bundle of linear frames over M with projection π. Denote by J (resp. J_0) the canonical complex structure of $\mathbf{R}^m$ [2, p. 115] (resp. $\mathfrak{gl}(m; \mathbf{R})$, the Lie algebra of the general linear group $GL(m; \mathbf{R})$, identified with the Lie algebra of all $m \times m$ real matrices). J is represented in the standard basis of $\mathbf{R}^m$ by the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the identity $n \times n$ matrix; J_0 acts on $\mathfrak{gl}(m; \mathbf{R})$ by left multiplication with the same matrix.

Suppose now we are given a linear connection Γ in P with ω as its 1-form of connection. Using the fact that Γ induces in each point $u \in P$ a direct sum decomposition of $T_u(P)$, the tangent space of P at u, into a vertical and a horizontal part, we make the following definition.

Definition 2.1. Let us call $\hat{J}$ the (1, 1)-tensor field on P given by

$$(2.1)\qquad \hat{J}X = \lambda J_0 \omega X + BJ\theta X \quad \forall u \in P,\ \forall X \in T_u(P),$$

where

(i) θ is the canonical $\mathbf{R}^m$-valued 1-form on P,

(ii) λ is the isomorphism between $\mathfrak{gl}(m; \mathbf{R})$ and the fundamental vector fields on P,

(iii) $B(\xi)$ is the standard horizontal vector field on P corresponding to $\xi \in \mathbf{R}^m$, whose value at u is the unique horizontal vector such that

$\pi(B(\xi)_u) = u(\xi)$, where u is interpreted as a linear isomorphism: $\mathbf{R}^m \to T_{\pi(u)}(M)$.

We have the following theorem.

Theorem 2.2. *If M is an even-dimensional smooth manifold with linear connection Γ, J (resp. J_0) the canonical complex structure on $\mathbf{R}^m$ (resp. on $\mathfrak{gl}(m; \mathbf{R})$), then:*

(1) *the (1, 1)-tensor field $\hat{J}$ defined on P by*

$$\hat{J} = \lambda J_0 \omega + BJ\theta$$

is an almost complex structure on P.

(2) *If $\omega' = \omega + \eta$ is another connection 1-form with η a tensorial form of type* adj GL$(m; \mathbf{R})$, *then the corresponding almost complex structure $\hat{J}'$ on P is given by*

$$\hat{J}' = \hat{J} + \lambda(J_0\eta - \eta BJ\theta).$$

(3) *If J_S (resp. J_Q) is a complex structure of $\mathbf{R}^m$ (resp. $\mathfrak{gl}(m; \mathbf{R})$) corresponding to the nontrivial left coset $[S]$ in GL$(m; \mathbf{R})/$GL$(n; \mathbf{C})$ (resp. $[Q]$ in GL$(m^2; \mathbf{R})/$GL$(2n^2; \mathbf{C})$), then the induced almost complex structure on P is given by*

$$\hat{J}_{Q,S} = -(\lambda J_Q J_0 \omega + BJ_S J\theta)\hat{J}.$$

Proof. (1) Because ω (resp. θ) is a vertical (resp. horizontal) form and $\theta B = 1$, we easily have $\hat{J}^2 = -1$.

(2) A standard horizontal vector field B' in the new connection $\omega' = \omega + \eta$ is given by $B' = B - \lambda\eta B$, because, for each $\xi \in \mathbf{R}^m$, $\eta B'(\xi) = -\omega B'(\xi)$ and $\omega\lambda + B\theta = 1$.

(3) By definition, $\hat{J}_{Q,S} = \lambda J_Q \omega + BJ_S\theta$, where $J_Q = QJ_0Q^{-1}$ and $J_S = SJS^{-1}$. The formula follows because $\omega\hat{J} = J_0\omega$ and $\theta\hat{J} = J\theta$.

From now on, we shall refer to the almost complex structure $\hat{J}$ defined above as the canonical almost complex structure corresponding to the condition ω or simply the *ω-canonical almost complex structure on P.*

3. **Complexifying connections.** It is known that a necessary and sufficient condition for an almost complex structure $\hat{J}$ to be integrable is the vanishing of its torsion $\hat{\tau}$ defined by

$$\hat{\tau}(X, Y) = [X, Y] + \hat{J}[\hat{J}X, Y] + \hat{J}[X, \hat{J}Y] - [\hat{J}X, \hat{J}Y], \tag{3.1}$$

where X and Y are vector fields on P.

Definition 3.1. For each $\alpha \in \Lambda^2(P, V)$, a 2-form on P with values in a vector space V with a complex structure J_V, let $\tilde{\alpha} \in \Lambda^2(P, V)$ defined by

$$-\tfrac{1}{2}\tilde{\alpha}(X, Y) = \alpha(X, Y) + J_V\alpha(\hat{J}X, Y) + J_V\alpha(X, \hat{J}Y) - \alpha(\hat{J}X, \hat{J}Y).$$

Theorem 3.2. *If Γ is a linear connection over M with ω as its 1-form of connection, then the torsion $\hat{\tau}$ of the ω-canonical almost complex structure $\hat{J}$ on P is given by*

$$\hat{\tau} = \lambda\tilde{\Omega} + B\tilde{\Theta}, \tag{3.2}$$

where Ω (resp. Θ) is the curvature form (resp. the torsion form of Γ).

Proof. Since each connection induces a parallelization of P by means of the vector fields $B_i = B(e_i)$, $1 \le i \le m$, where $\{e_1, \cdots, e_m\}$ is the standard basis of $\mathbf{R}^m$, and the vector fields $X_i^j = \lambda E_i^j$, $1 \le i, j \le m$, where $(E_i^j)_{1 \le i,j \le m}$ is the basis of $\mathfrak{gl}(m; \mathbf{R})$ defined by $(E_i^j)_l^k = \delta_l^j \delta_i^k$, it is sufficient to compute the torsion in the three following cases:

(a) X and Y are horizontal; more precisely $X = B_1$ and $Y = B_2$, where $B_i = B(\xi_i)$, ξ_1 and ξ_2 linearly independent in $\mathbf{R}^m$.

(b) X is horizontal; $X = B(\xi)$ for ξ nonzero in $\mathbf{R}^m$ and Y is vertical, say $Y = \lambda A$ where A nonzero in $\mathfrak{gl}(m, \mathbf{R})$.

(c) X and Y are vertical; $X = \lambda A_1$ and $Y = \lambda A_2$ with A_1 and A_2 linearly independent in $\mathfrak{gl}(m; \mathbf{R})$.

Using the structure equations [1, p. 120] for Ω and Θ, we first get $[B_1, B_2] = -2\lambda\Omega(B_1, B_2) - 2B\Theta(B_1, B_2)$. Secondly by [1, p. 119] we have $[B(\xi), \lambda A] = -B(A\xi)$, and finally, $[\lambda A_1, \lambda A_2] = \lambda[A_1, A_2]$.

Observing that $\hat{J}B(\xi) = B(J\xi)$ since $\theta B = 1$, and $\hat{J}\lambda A = \lambda J_0 A$ since $\omega\lambda = 1$, the theorem follows by computation and use of Definition 2.1.

This result suggests calling a *complexifying connection* a connection whose curvature and torsion satisfy the conditions $\tilde{\Omega} = 0$ and $\tilde{\Theta} = 0$. Then Theorem 3.2 can be stated in the following way:

Theorem 3.2′. *The ω-canonical almost complex structure on P induced by a connection Γ is integrable if and only if Γ is a complexifying connection.*

A flat linear connection on M, without torsion, gives a (rather trivial) example of such a connection, since $\Omega = 0$ (resp. $\Theta = 0$) implies $\tilde{\Omega} = 0$ (resp. $\tilde{\Theta} = 0$). However, it is important to note that, in general, the form $\tilde{\alpha}$ can be zero without α being zero. More precisely we have the following theorem.

Let us first call *mixed torsion map* the endomorphism μ of $\Lambda^2(P, V)$ defined by $\mu\alpha = -\frac{1}{2}\tilde{\alpha}$. The name is suggested by the formal resemblance to Definition 3.1 of $\tilde{\tau}$, the structures J_V and $\hat{J}$ being "mixed" together, and also by formula (3.2).

Theorem 3.3. *The mixed torsion map μ is singular.*

Proof. The fact that μ is linear is trivial. To show that the kernel of μ is different from zero, let us first note that $\mu^2 - 4\mu = 0$ by an easy computation. On the other hand, $\mu \neq 4I$ by the Lemma below (whose proof was kindly suggested to the author by Professor S. Takahashi). Suppose now that μ is nonsingular, then $(\mu - 4)\alpha$ is in the kernel of μ for each α, hence so is $\mu\alpha = 4\alpha$, which gives a contradiction. Q.E.D.

Lemma. *For the mixed torsion map μ we have $\mu \neq 4I$.*

Proof. Write $V = V_0 \oplus J_V V_0$, viewing V as a complex space and V_0 as the set of $\alpha \in V$ such that $\bar{\alpha} = \alpha$ (bar denotes complex conjugation). Accordingly we write $\alpha = \alpha^+ + J_V\alpha^-$ for each $\alpha \in V$. It is sufficient to show that $\mu = 4I$ and $\alpha^- = 0$ implies $\alpha^+ = 0$. By definition of μ, we have

$$3\alpha(X, Y) = J_V\alpha(\hat{J}X, Y) + J_V\alpha(X, \hat{J}Y) - \alpha(\hat{J}X, \hat{J}Y).$$

Thus,

$$3\alpha^+(X, Y) = -\alpha^+(\hat{J}X, \hat{J}Y) \quad \text{and} \quad \alpha^+(X, Y) = \alpha^+(\hat{J}X, \hat{J}Y).$$

Hence $4\alpha^+(X, Y) = 0$. Q.E.D.

For low-dimensional manifolds we have the following result.

Theorem 3.4. *Each linear connection on a 2-dimensional manifold is a complexifying connection.*

Proof. Take any nonzero element in $\mathbf{R}^2$ and call B the standard horizontal vector field corresponding to it. By Theorem 3.2, it is sufficient to show that $\tilde{\alpha}(B, \hat{J}B) = 0$, where $\alpha = \Omega$ (resp. Θ), because these are horizontal 2-forms. By the skew symmetry of α and Definition 3.1 we have

$$-\tfrac{1}{2}\tilde{\alpha}(B, \hat{J}B) = \alpha(B, \hat{J}B) + J_V\alpha(\hat{J}B, \hat{J}B) - J_V\alpha(B, B) + \alpha(\hat{J}B, B) = 0. \quad \text{Q.E.D.}$$

Remark. Using the parallelization of P given by a connection Γ, we can introduce a Riemannian metric on P in a standard way, and it is easy to verify that with respect to this metric, Γ induces an almost hermitian structure on P, which is hermitian in case Γ is a complexifying connection. However, such an (almost) hermitian structure is never (almost) Kählerian, as one can easily check.

4. Almost complex structures, up and down. From now on, we shall assume that M, itself, admits an almost complex structure J_M. It is known that the existence of an almost complex structure J_M on M is equivalent to the reduction of $L(M)$ to $C(M)$, the bundle of complex linear frames, the subbundle of $L(M)$ defined by $C(M) = \{u \in L(M);\ uJ = J_M u\}$. (We view u as in Definition 2.1(iii).) Let ω be any linear connection on M. If X is a vector field on X and X^* its horizontal lift to $L(M)$, we can compare $\hat{J}X^*$ with $(J_M X)^*$ in the following theorem.

Theorem 4.1. *For each vector field X on M, we have $\hat{J}X^* = (J_M X)^*$ on $C(M)$.*

Proof. For an arbitrary point u in $L(M)$ we have by definition of $\hat{J}$, $(\hat{J}X^*)_u = (BJ\theta(X_u^*))_u = Y$, the unique horizontal vector at u such that $\pi Y = uJ\theta(X_u^*) = uJu^{-1}X_{\pi(u)}$. On the other side $(J_M X)_u^* = Z$, the unique horizontal vector at u such that $\pi Z = J_M X_{\pi(u)}$. Therefore

$$Y = Z \Leftrightarrow uJu^{-1}X_{\pi(u)} = J_M X_{\pi(u)} \Leftrightarrow uJu^{-1} = J_M \Leftrightarrow u \in C(M). \quad \text{Q.E.D.}$$

From now on we shall assume that ω is the 1-form of an almost complex connection, i.e. a linear connection arising from a connection on $C(M)$. This is known to be equivalent with the fact that J_M is parallel with respect to this connection. Still denoting by $\hat{J}$ the corresponding almost complex structure on $L(M)$, we have the following relationship between its torsion $\hat{\tau}$ and τ_M, the torsion of J_M.

Theorem 4.2. *If (M, J_M) is an almost complex manifold, and if ω is the 1-form of an almost complex connection on M, then*

$$B\tilde{\Theta}(X^*, Y^*) = (\tau_M(X, Y))^* \tag{4.1}$$

on $C(M)$, where X and Y are vector fields on M.

Proof. According to (3.2) the torsion of $\hat{J}$ is given by $\hat{\tau}(X^*, Y^*) = \lambda\tilde{\Omega}(X^*, Y^*) + B\tilde{\Theta}(X^*, Y^*)$. We are interested only in the second term of the right member, the first one being in the kernel of the differential of π. Formula (3.1) gives

$$\hat{\tau}(X^*, Y^*) = [X^*, Y^*] + \hat{J}[\hat{J}X^*, Y^*] + \hat{J}[X^*, \hat{J}Y^*] - [\hat{J}X^*, \hat{J}Y^*].$$

If we restrict ourselves to $C(M)$ then, by Theorem 2.2 and the very definition of an almost complex connection, the horizontal part of the right member of this equation becomes

$$[X, Y]^* + (J_M[J_M X, Y])^* + (J_M[X, J_M Y])^* - [J_M X, J_M Y]^*,$$

which is exactly the horizontal lift of the vector field $\tau_M(X, Y)$. Q.E.D.

This theorem has two interesting and simple applications. The first one is the converse (in the case of almost complex structures) of the general theorem [3], that if a G-structure on M is integrable it admits a torsionfree connection. We have

Corollary 4.3. *If an almost complex manifold (M, J_M) admits a torsion-free almost complex connection, then J_M is integrable.*

Proof. $\Theta = 0$ implies $\tilde{\Theta} = 0$ and, by Theorem 4.2, $\tau_M = 0$.

The second application is a geometric interpretation of the identity (1.1) which is simply the projection down to M of the torsion of $\hat{J}$ evaluated on horizontal lifts in $C(M)$. This is a consequence of the definition of the torsion T of the connection as $T(X, Y) = u2\Theta(X^*, Y^*)$ and of $\tilde{\Theta}$ as

$$-\tfrac{1}{2}\tilde{\Theta}(X^*, Y^*) = \Theta(X^*, Y^*) + J\Theta(\hat{J}X^*, Y^*) + J\Theta(X^*, \hat{J}Y^*) - \Theta(\hat{J}X^*, \hat{J}Y^*).$$

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