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IDENTITIES FOR CONJUGATION IN THE STEENROD ALGEBRA

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ABSTRACT. Let χ be the canonical conjugation in the Steenrod algebra $\mathcal{A}_2$. I prove the identity

$$Sq^{2^n} + \chi(Sq^{2^n}) = Sq^{2^{n-1}}\chi(Sq^{2^{n-1}})$$

and generalizations of this identity both in $\mathcal{A}_2$ and in $\mathcal{A}_p$ where p is an odd prime.

The canonical conjugation χ in the mod 2 Steenrod algebra $\mathcal{A}_2$ can be defined by Thom's recursion formula

$$\sum_{i=0}^{n} Sq^i\chi(Sq^{n-i}) = 0$$

together with the stipulation that $\chi\colon \mathcal{A}_2 \to \mathcal{A}_2$ be an anti-isomorphism [3]. Since the elements Sq^{2^n} multiplicatively generate $\mathcal{A}_2$, we can calculate χ if we can calculate $\chi(Sq^{2^n})$ for all n. The above recursion formula is unnecessarily cumbersome for this goal. In fact, the recursion can be shortened considerably by use of the following interesting

Identity. $Sq^{2^n} + \chi(Sq^{2^n}) = Sq^{2^{n-1}}\chi(Sq^{2^{n-1}})$ for all positive integers n.

Applying the identity recursively we obtain the

Formula.

$$\chi(Sq^{2^n}) = Sq^{2^n} + \sum_{i=1}^{n-1}\left(\prod_{j=1}^{i} Sq^{2^{n-j}}\right)Sq^{2^{n-i}}.$$

For example,

$$\chi(Sq^{16}) = Sq^{16} + Sq^8Sq^8 + Sq^8Sq^4Sq^4 + Sq^8Sq^4Sq^2Sq^2.$$

In this paper, I will prove a theorem which will imply the above identity, and which also yields results about the mod p Steenrod algebra $\mathcal{A}_p$ when p is an odd prime. The technique is to use Milnor's calculation of χ in the

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Milnor basis for $\mathcal{A}_p$ [2], together with properties of binomial coefficients mod p. This technique was noticed independently by Donald Davis, who used it to prove other identities involving χ in $\mathcal{A}_p$ [1].

Let p be a fixed prime. Let $R = (r_1, r_2, \cdots)$ be a sequence of nonnegative integers with only a finite number of nonzero terms. For each such R there is an element $\mathcal{P}^R$ in the Milnor basis for $\mathcal{A}_p$, of degree $\Sigma_{i\geq 1} 2(p^i - 1)r_i$ (if $p = 2$, the element is written Sq^R and its degree is $\Sigma_{i\geq 1}(2^i - 1)r_i$). We define $|R| = \Sigma_{i\geq 1} p^{i-1} r_i$. Then Davis' main proposition can be written.

Proposition 1. $\mathcal{P}^m \chi(\mathcal{P}^n) = (-1)^n \Sigma_R \binom{|R|}{m} \mathcal{P}^R$ *where the sum is taken over all* R *such that* $\mathcal{P}^R$ *has the proper degree, i.e. over all* R *such that*

$$\sum_{i\geq 1} (p^i - 1)r_i = (p-1)(m+n).$$

If $p = 2$, the only necessary modification is to write Sq for $\mathcal{P}$. The binomial coefficient is, of course, to be interpreted mod p.

Proof. See [1]. □

The one additional fact about binomial coefficients which we will need is

Proposition 2. *Let* a *and* b *be integers. If* $p^a \leq r \leq p^a b$, *then*

$$\sum_{k=0}^{b} (-1)^k \binom{r}{p^a k} \equiv 0 \pmod{p}.$$

Proof. Write $r = p^a s + t$, with $1 \leq s \leq b$ and $0 \leq t < p^a$. Then

$$\binom{p^a s + t}{p^a k} \equiv \binom{s}{k} \pmod{p},$$

as is easily seen by comparing the coefficients of $x^{p^a k}$ in the congruence

$$(1+x)^{p^a s + t} \equiv (1 + x^{p^a})^s (1+x)^t \pmod{p}.$$

Hence the proposition follows from the well-known identity $\Sigma_{k=0}^b (-1)^k \binom{s}{k} = 0$ for $1 \leq s \leq b$. □

We can now prove our main

Theorem. *Let* $a \geq 0$ *and* $b > 1$ *be integers. Then*

$$\sum_{k=0}^{b} \mathcal{P}^{p^a k} \chi(\mathcal{P}^{p^a(b-k)}) = 0.$$

Examples. (1) If $a = 0$, we get Thom's original recursion formula in $\mathcal{A}_p$.

(2) If $p = 2$, $a = n - 1$, $b = 2$, we get the identity at the beginning of this paper.

(3) If $p = 2$, $a = 2$, $b = 3$, we get

$$\chi(Sq^{12}) + Sq^4\chi(Sq^8) + Sq^8\chi(Sq^4) + Sq^{12} = 0.$$

(4) If $p = 3$, $a = 2$, $b = 3$, we get

$$\chi(\mathcal{P}^{27}) + \mathcal{P}^9\chi(\mathcal{P}^{18}) + \mathcal{P}^{18}\chi(\mathcal{P}^9) + \mathcal{P}^{27} = 0.$$

Proof of Theorem. Consider any $R = (r_1, r_2, \cdots)$ such that

$$(*) \qquad \sum_{i \geq 1} (p^i - 1)r_i = (p - 1)p^a b.$$

The coefficient of $\mathcal{P}^R$ in the Milnor base expansion of the sum in the Theorem is, by Proposition 1,

$$\sum_{k=0}^{b} (-1)^{p^a(b-k)} \binom{|R|}{p^a k} = (-1)^b \sum_{k=0}^{b} (-1)^k \binom{|R|}{p^a k}.$$

By Proposition 2, this coefficient is zero if $p^a \leq |R| \leq p^a b$. But (*) gives that

$$|R| = \sum_{i \geq 1} p^{i-1} r_i = \frac{1}{p}\left[(p - 1)p^a b + \sum_{i \geq 1} r_i\right]$$

and we also have

$$0 \leq \sum_{i \geq 1} r_i \leq \frac{1}{p - 1} \sum_{i \geq 1} (p^i - 1)r_i = p^a b.$$

Hence $((p - 1)/p)p^a b \leq |R| \leq p^a b$; and since $b > 1$, the required inequality holds. □

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