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ON INFINITE DIMENSIONAL ITO'S FORMULA

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ABSTRACT. Ito's formula for $Y(t)g(t, \xi(t))$ is given where g is a Hilbert space valued function, $\xi(t)$ is a diffusion on Hilbert space and $Y(t)$ is an operator-valued stochastic integral w.r.t. $\xi(t)$. A stochastic integral representation for solution of a certain second order parabolic equation is also given.

0. **Introduction.** Infinite dimensional Ito's formulas have been studied in [1] and [3]. The former is for vector-valued functions and S-Wiener processes on Hilbert space, where S is an S-operator (cf. [5]); that is, a selfadjoint, nonnegative, trace class operator, on the underlying Hilbert space. The latter is for operator-valued functions and abstract Wiener processes on abstract Wiener space. An S-Wiener process is an abstract Wiener process. An abstract Wiener space can be extended to a Hilbert space with some Gaussian measure associated with some S-operator S (cf. [2]). Therefore, the notions of S-Wiener process and abstract Wiener process are equivalent. In this note, the underlying process will satisfy $d\xi(t) = A(t, \xi(t))db(t)$, where A is an operator-valued function and $b(t)$ is an S-Wiener process on Hilbert space. The main purpose is to study Ito's formula for $Y(t)g(t, \xi(t))$, where g is a Hilbert space valued function and $Y(t)$ is an operator-valued stochastic integral w.r.t. $\xi(t)$. As an application, the stochastic integral representation for solution of a certain infinite-dimensional parabolic equation is given.

1. **Notations.** Throughout this note, G, H, K will denote real, separable Hilbert spaces with complete orthonormal basis $\{u_n\}, \{v_n\}$ and $\{w_n\}$ respectively. Let $\mathfrak{L}_0, \mathfrak{L}_2$ denote the standard notations for the space of bounded linear (or bilinear) operators from Hilbert spaces to Hilbert spaces with operator norm $|||\cdot|||_{op}$ and Hilbert-Schmidt norm $|||\cdot|||_{HS}$ respectively.

For a trace class operator T, let trace T denote the ordinary trace of T. If $T \in \mathfrak{L}_0(H^2, K)$, let

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$$\text{Trace } T = \sum_{n=1}^{\infty} T(v_n, v_n)$$

when $\Sigma_{n=1}^{\infty} T(v_n, v_n)$ converges absolutely. This definition is independent of the basis chosen in H. Similarly, one can define Trace T for $T \in \mathfrak{L}_0(G^2, K)$. If $Y \in \mathfrak{L}_0(K)$ and Trace B exists, then

$$\text{Trace } Y \circ B = Y(\text{Trace } B). \tag{1.1}$$

Suppose that B is a bilinear operator and A, C are linear operators. Define

$$B \circ (A, C)(x, y) = B(Ax, Cy),$$
$$(B \,\Delta\, C)(x, y) = B(x, Cy)$$

whenever the right sides of the above definitions are meaningful. If $B \in \mathfrak{L}_0(H^2, K)$, $A \in \mathfrak{L}_0(G, H)$ and $(A^*)^{-1}$ exists, then it can be checked that

$$\text{Trace } B = \text{Trace } B \circ ((A^*)^{-1}, A). \tag{1.2}$$

For $x \in H$, $y \in K$, let $y \otimes x$ denote the element in $\mathfrak{L}_0(H, K)$ such that

$$(y \otimes x)(z) = \langle z, x\rangle_H y$$

for $z \in H$. It is easily seen that $|||y \otimes x|||_{HS} = \|y\|_K \|x\|_H$, $(y \otimes x)^* = x \otimes y$ and that $\{w_n \otimes v_m\}$ is a basis for $\mathfrak{L}_0(H, K)$.

Suppose that g is a K-valued function on H; then the first and second order Fréchet derivatives of g at $x \in H$ will be denoted by $Dg(x)$ and $D^2g(x)$ respectively. If we express $g(x) = \Sigma_{n=1}^{\infty} g_n(x) w_n$, then

$$Dg(x) = \sum_{n=1}^{\infty} w_n \otimes Dg_n(x) \tag{1.3}$$

when $Dg(x)$ exists, where $Dg_n(x) \in \mathfrak{L}_0(H, R) \cong H$.

2. **Stochastic integrals and Ito's formula.** Let S be a selfadjoint, positive definite, trace class operator on G with eigenvalues λ_n, $n \geq 1$. Since the eigenvectors of S form a basis of G, it will be assumed that u_n, $n \geq 1$, are the eigenvectors of S corresponding to eigenvalues λ_n, $n \geq 1$, respectively. Let $b(t)$, $t \geq 0$, be an S-Wiener process on G (cf. [1]), that is with probability one, $b(0) = 0$, $b(t)$ is t-continuous on G, and $b(t)$ has independent increment and satisfies

$$E\{\exp(i\langle x, b(t)\rangle_G) | \mathfrak{M}_0^s\} = \exp\left\{i\langle x, b(s)\rangle_G - \frac{t-s}{2}\langle Sx, x\rangle_G\right\}$$

for $0 \leq s \leq t$ and $x \in G$, where $\mathfrak{M}_r^s = \sigma\{b(u): r \leq u \leq s\}$ for $0 \leq r \leq s$. Note that $b(t)$ can be expressed as

$$b(t) = \sum_{n=1}^{\infty} b_n(t)u_n$$

where $b_n(t)$, $n \geq 1$, are independent 1-dimensional Wiener processes with parameters λ_n, $n \geq 1$, respectively. For an $\mathfrak{L}_0(G, H)$-valued s-n.a. (s-nonanticipating) function $A(t)$ with $\int_s^t E|||A(r)|||_{\mathrm{op}}^2 dr < \infty$ for $t \geq s$, the H-valued stochastic integral $\int_s^t A(r)db(r)$ can be defined as usual for $t \geq s$ and that $E\{\int_s^t A(r)db(r)\} = 0$ and

$$E\left\{\left\|\int_s^t A(r)db(r)\right\|_H^2\right\} = \int_s^t E|||A(u) \circ \sqrt{S}\,|||_{\mathrm{HS}}^2 dr$$

$$\leq \operatorname{trace} S \int_s^t E|||A(r)|||_{\mathrm{op}}^2 dr$$

where $\sqrt{S}$ is the positive square root of S.

Let M be a positive constant and $A(t, x)$ be an $\mathfrak{L}_0(G, H)$-valued measurable function on $[0, \infty) \times H$ such that

$$|||A(t, x) - A(t, y)|||_{\mathrm{op}} \leq M\|x - y\|_H,$$

$$|||A(t, x)|||_{\mathrm{op}} \leq M(1 + \|x\|_H)$$

for $t \geq 0$, $x, y \in H$. Then for each $x \in H$, the stochastic integral equation

$$\xi(t) = x + \int_s^t A(r, \xi(r))db(r)$$

has a unique continuous solution $\xi(t) = \xi_{s,x}(t)$, $t \geq s$. Suppose that $B(t)$ is an $\mathfrak{L}_0(H, \mathfrak{L}_2(K))$-valued s-n.a. function that satisfies the condition

$$(2.1)\qquad \begin{cases} \int_s^t E|||B(r)|||_{\mathrm{op}}^4 dr < \infty & \text{or} \\ \int_s^t E|||B(r)|||_{\mathrm{op}}^2 dr < \infty & \text{if } A(t, x) \text{ is bounded,} \end{cases}$$

for $t \geq s$; then $\int_s^t B(r)d\xi(r)$ can be defined for $t \geq s$ and

$$(2.2)\qquad E\left\{\left|\left|\left|\int_s^t B(r)d\xi(r)\right|\right|\right|_{\mathrm{HS}}^2\right\} = \int_s^t E|||B(r) \circ A(r, \xi(r)) \circ \sqrt{S}|||_{HS}^2 dr$$

$$\leq \operatorname{trace} S \int_s^t E|||B(r) \circ A(r, \xi(r))|||_{\mathrm{op}}^2 dr$$

where $B(r) \circ A(r, \xi(r)) \circ \sqrt{S} \in \mathfrak{L}_2(G, \mathfrak{L}_2(K))$.

Let $C_b^{1,2}$ denote the class of all K-valued functions on $[0, \infty) \times H$ which have bounded continuous partial derivative w.r.t. t and bounded continuous first and second order Fréchet derivatives w.r.t. $x \in H$.

Theorem 1 (Ito's formula). *Let* $B(r) \in \mathcal{L}_0(H, \mathcal{L}_2(K))$ *satisfy condition* (2.1) *and* $C(r) \in \mathcal{L}_2(K)$ *satisfy* $\int_s^t E\|\|C(r)\|\|_{HS}^2 dr < \infty$ *for* $t \geq s$. *For* $F \in \mathcal{L}_0(K)$, *let*

$$Y(t) = F + \int_s^t B(r)d\xi(r) + \int_s^t C(r)\,dr, \qquad t \geq s.$$

Then for $g \in C_b^{1,2}$,

$$\begin{aligned}
&Y(t)g(t, \xi(t)) - Y(s)g(s, \xi(s)) \\
&= \int_s^t \{Y(r) \circ Dg(r, \xi(r)) + B(r)(\cdot, g(r, \xi(r)))\}d\xi(r) \\
&\quad + \int_s^t \Bigg\{ Y(r)\Bigg[\frac{\partial}{\partial r} g(r, \xi(r)) \\
&\qquad + \text{Trace } D^2 g(r, \xi(r)) \circ (A(r, \xi(r)), A(r, \xi(r))) \circ (\sqrt{S}, \sqrt{S})\Bigg] \\
&\qquad + \text{Trace}\,[B(r) \,\Delta\, Dg(r, \xi(r))] \circ (A(r, \xi(r)) \circ S,\ A(r, \xi(r))) \\
&\qquad + C(r)g(r, \xi(r))\Bigg\} dr
\end{aligned} \tag{2.3}$$

where $B(\cdot, x)$ *is defined as* $B(\cdot, x)y = B(y, x)$ *for* $x \in K$, $y \in H$.

Sketch of proof. For each $n \geq 1$, let

$$\xi^{(n)}(t) = x + \int_s^t A(r, \xi(r))db^{(n)}(r) = x + \sum_{k=1}^{n} \int_s^t A(r, \xi(r))u_k db_k(r),$$

$$g^{(n)}(t, \xi(t)) = g(t, \xi^{(n)}(t)) = \sum_{k=1}^{\infty} g_k^{(n)}(t, \xi(t)) = \sum_{k=1}^{\infty} g_k(t, \xi^{(n)}(t)),$$

$$Y^{(n)}(t) = F + \int_s^t B(r)d\xi^{(n)}(r) + \int_s^t C(r)\,dr$$

where $b^{(n)}(t)$ is the projection of $b(t)$ on the span of u_k, $1 \leq k \leq n$. Then the stochastic differentials of $g_k^{(n)}(t, \xi(t))$, $1 \leq k \leq n$, are

$$\begin{aligned}
dg_k^{(n)}(t) &= \langle Dg_k^{(n)}(t), d\xi^{(n)}(t)\rangle_H \\
&\quad + \left\{\frac{\partial}{\partial t} g_k^{(n)}(t) + \frac{1}{2}\text{trace } D^2 g_k^{(n)}(t) \circ (A(t), A(t)) \circ (\sqrt{S_n}, \sqrt{S_n})\right\} dt
\end{aligned}$$

where

$$S_n(u_m) = \begin{cases} \lambda_m u_m, & m \leq n, \\ 0, & m > n, \end{cases}$$

and where $g_k^{(n)}(t) = g_k^{(n)}(t, \xi(t))$ and $A(t) = A(t, \xi(t))$.

Let $y \in K$. The stochastic differentials for the product of $g_k^{(n)}(t)$ and $\langle Y^{(n)}(t)w_k, y\rangle_K$, $1 \le k \le n$, are

$$
\begin{aligned}
&dg_k^{(n)}(t)\langle Y^{(n)}(t)w_k, y\rangle_K \\
&\quad = \langle \{Y^{(n)}(t) \circ (w_k \otimes Dg_k^{(n)}(t)) + B(\cdot, g_k^{(n)}(t)w_k)\} d\xi^{(n)}(t), y\rangle_K \\
&\qquad + \left\langle Y^{(n)}(t)\left\{\frac{\partial}{\partial t} g_k^{(n)}(t) + \frac{1}{2}\operatorname{trace} D^2 g_k^{(n)}(t) \circ (A(t), A(t)) \circ (\sqrt{S_n}, \sqrt{S_n})\right\} w_k \right. \\
&\qquad \left. + C(t)(g_k^{(n)}(t)w_k) + \operatorname{Trace}[B(t)\,\Delta\,(w_k \otimes Dg_k^{(n)}(t))] \circ (A(t) \circ S_n, A(t)), y\right\rangle_K dt.
\end{aligned}
$$

After summing the above identities over $1 \le k \le n$, applying identity (1.3) and using the fact that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \operatorname{trace} D^2 g_k^{(n)}(t) \circ (A(t), A(t)) \circ (\sqrt{S_n}, \sqrt{S_n}) w_k \\
&\qquad\qquad = \operatorname{Trace} D^2 g^{(n)}(t) \circ (A(t), A(t)) \circ (\sqrt{S_n}, \sqrt{S_n}),
\end{aligned}
$$

where $g^{(n)}(t) = g^{(n)}(t, \xi(t))$, one obtains

$$
\begin{aligned}
&\langle dY^{(n)}(t) g^{(n)}(t), y\rangle_K \\
&\quad = \langle \{Y^{(n)}(t) \circ Dg^{(n)}(t) + B(t)(\cdot, g^{(n)}(t))\} d\xi^{(n)}(t), y\rangle_K \\
&\qquad + \left\langle Y^{(n)}(t)\left\{\frac{\partial}{\partial t} g^{(n)}(t) + \frac{1}{2}\operatorname{Trace} D^2 g(t) \circ (A(t), A(t)) \circ (\sqrt{S_n}, \sqrt{S_n})\right\} \right. \\
&\qquad\qquad \left. + \operatorname{Trace}(B(t)\,\Delta\, Dg(t)) \circ (A(t)S_n, A(t)) + C(t) g^{(n)}(t), y\right\rangle_K.
\end{aligned}
$$

The assumptions on $A(t, x)$, $B(t)$ and $C(t)$ permit letting $n \to \infty$ in the above identity. Identity (2.3) then follows.

Theorem 2. *Let $B(r)$ be an $\mathfrak{L}_0(H, \mathfrak{L}_2(K))$-valued s.n.a. function, and let $C(r)$ be an $\mathfrak{L}_2(K)$-valued s.n.a. function such that*

$$|||B(r) \circ A(r, \xi(r)) \circ \sqrt{S}|||_{HS}$$

is bounded and $\int_s^t E|||C(r)|||_{HS}^2 dr < \infty$ for $t \ge s$. Then for $F \in \mathfrak{L}_0(K)$, the $\mathfrak{L}_0(K)$-valued stochastic integral equation

$$Y(t) = F + \int_s^t Y(r) \circ B(r) d\xi(r) + \int_s^t Y(r) \circ C(r)\, dr \tag{2.4}$$

has a unique continuous solution.

Sketch of proof. Define $Y_0(t) \equiv F$ and inductively

$$Y_n(t) = F + \int_s^t Y_{n-1}(t) \circ B(r) d\xi(r) + \int_s^t Y_{n-1}(r) \circ C(r)\, dr$$

for $n \geq 1$, $t \geq s$. Using the assumptions on $B(r)$, $C(r)$ and inequality (2.2), one obtains

$$E|||Y_n(t) - Y_{n-1}(t)|||^2_{\mathrm{HS}} \leq \frac{N^n(t-s)^n}{n!}, \tag{2.5}$$

where N is a constant. Apply the submartingale inequality to the submartingale

$$\{|||Y_{n+1}(t) - Y_n(t)|||^2_{\mathrm{HS}}, \mathfrak{M}^t_s;\ t \geq s\};$$

then use inequality (2.5) and the Borel-Cantelli lemma to show that

$$\sum_{n=0}^{\infty} (Y_{n+1}(t) - Y_n(t))$$

converges in $|||\cdot|||_{\mathrm{HS}}$ norm uniformly in t-compact sets a.e. (cf. [4]). Therefore,

$$Y(t) = \lim_{n\to\infty} Y_n(t) = Y_0 + \sum_{n=0}^{\infty} (Y_{n+1}(t) - Y_n(t))$$

converges in $|||\ |||_{\mathrm{op}}$ norm uniformly in t-compact sets a.e. Similar argument shows that $Y(t)$ is the unique solution of equation (2.4).

Note that when we substitute $Y(t)$ of (2.4) into identity (2.3) and use identity (1.1) Ito's formula becomes

$$\begin{aligned} &Y(t)g(t, \xi(t)) - Y(s)g(s, \xi(s)) \\ &= \int_s^t Y(r)\{Dg(r, \xi(r)) + B(r)(\cdot\,, g(r, \xi(r)))\}d\xi(r) \\ &+ \int_s^t Y(r)\Big\{\frac{\partial}{\partial r} g(r, \xi(r)) \\ &\qquad + \operatorname{Trace} D^2 g(r, \xi(r)) \circ (A(r, \xi(r)), A(r, \xi(r))) \circ (\sqrt{S}, \sqrt{S}) \\ &\qquad + \operatorname{Trace}[B(r) \,\Delta\, Dg(r, \xi(r))] \circ (A(r, \xi(r)) \circ S, A(r, \xi(r))) \\ &\qquad + C(r)g(r, \xi(r))\Big\} dr. \end{aligned} \tag{2.6}$$

3. **Application.** Suppose that the $\mathfrak{L}_0(G, H)$-valued function $A(t, x)$ satisfies condition

$$M_1\|y\|_G \le \|A(t, x)y\|_H \le M_2\|y\|_G$$

for $0 \le t \le T$, $x \in H$ and $y \in G$, where $0 < M_1 \le M_2$ are constants. Let $B(t, x)$ be a bounded, $\mathcal{L}_0(H, \mathcal{L}_2(K))$-valued measurable function, $C(t, x)$ be an $\mathcal{L}_2(K)$-valued measurable function on $[0, T] \times H$ such that

$$|||C(t, x)|||_{\mathrm{op}} \le \text{constant}(1 + \|x\|_H).$$

Consider the second order equation

$$\tag{3.1} \begin{aligned} &\frac{\partial}{\partial t} f(t, x) + \frac{1}{2}\operatorname{Trace}\{D^2 f(t, x) \circ (A(t, x), A(t, x)) \circ (\sqrt{S}, \sqrt{S})\} \\ &\qquad + \operatorname{Trace}(B(t, x) \vartriangle Df(t, x)) + C(t, x)f(t, x) + g(t, x) = 0 \end{aligned}$$

for $x \in H$, $0 < t < T$, with initial condition $f(T, x) = h(x)$, where f, g and h are K-valued functions. In the case that $G = H = R^d$, $K = R^N$ and $S = I_d$ (the identity on R^d), the equation (3.1) is reduced to the standard second order parabolic equation (cf. [6]). In order that equation (3.1) be meaningful, a condition must be imposed on $B(t, x)$ so that Trace$(B(t, x) \vartriangle Df(t, x))$ exists.

For each $n \ge 1$, let an operator S_n' on G be defined as

$$S_n'(u_m) = \begin{cases} \lambda_m^{-1} u_m, & m \le n, \\ 0, & m > n. \end{cases}$$

Assume that $B(t, x) \circ A^*(t, x)^{-1} \circ S_n'$ converges strongly to a bounded limit function denoted by $B(t, x) \circ A^*(t, x)^{-1} \circ S^{-1}$. It is clear that

$$B(t, x) \circ A^*(t, x)^{-1} = (B(t, x) \circ A^*(t, x)^{-1} \circ S^{-1}) \circ S$$

and hence

$$\begin{aligned} &\|\operatorname{Trace}(B(t, x) \vartriangle Df(t, x)) \circ (A^*(t, x)^{-1}, A(t, x))\|_K \\ &\quad \le \sum_{n=1}^{\infty} \|B(t, x)(A^*(t, x)^{-1} u_n, Df(t, x)A(t, x)u_n)\|_K \\ &\quad \le \operatorname{trace} S |||B(t, x) \circ A^*(t, x)^{-1} \circ S^{-1}|||_{\mathrm{op}} |||Df(t, x)|||_{\mathrm{op}} |||A(t, x)|||_{\mathrm{op}} < \infty. \end{aligned}$$

Therefore, by identity (1.2), Trace$(B(t, x) \vartriangle Df(t, x))$ exists.

Under the above assumptions on $A(t, x)$, $B(t, x)$ and $C(t, x)$, the $\mathcal{L}_0(K)$-valued stochastic integral equation

$$\tag{3.2} \begin{aligned} Y(t) = I &+ \int_s^t Y(r) \circ (B(r, \xi(r)) \circ A^*(r, \xi(r))^{-1} \circ S^{-1}) \circ A(r, \xi(r))^{-1} d\xi(r) \\ &+ \int_s^t Y(r) \circ C(r, \xi(r))\, dr \end{aligned}$$

has a unique solution $Y_s(t)$, $0 \le s \le t \le T$. If $f \in C_b^{1,2}$ is a solution of equation (3.1), then, by substituting (3.2) into identity (2.6) and using equation (3.1) and initial condition $f(T, x) = h(x)$, one obtains the stochastic integral representation

$$f(s, x) = E\{Y_s(T)h(\xi(T))\} + E\left\{\int_s^T Y_s(r)g(r, \xi(r))\,dr\right\}$$

with

$$\xi(t) = x + \int_s^t A(r, \xi(r))db(r), \quad t \geq s.$$

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