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CHARACTERIZATIONS OF THE COMPLEX PROJECTIVE PLANE BY CURVATURE

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ABSTRACT. We give short proofs to show that under various positivity assumptions on the curvature of a Kähler surface X, it is biholomorphically equivalent to $P_2(C)$. In particular, the case of δ-holomorphic pinching $>\frac{1}{2}$ (Theorem 1) is best possible and, we believe, new.

Let X be a compact Kähler manifold of complex dimension 2. Given a real two-dimensional subspace p of the real tangent space of X at some point, we denote by $K(p)$ the Riemannian curvature of p. X is called *δ-homomorphically pinched* if there is some constant $A > 0$ such that $\delta A \leq K(p) \leq A$ for all holomorphic planes p, i.e. for all planes p invariant under the complex structure endomorphism J. Given two holomorphic planes

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p and p', the holomorphic bisectional curvature $H(p, p')$ associated to p and p' is defined to be $R(X, JX, Y, JY)$ where X and Y are unit vectors in p and p', respectively, and R is the Riemannian curvature tensor. For further details see Kobayashi and Nomizu [11].

Theorem 1. *If X is δ-homomorphically pinched, $\delta > \frac{1}{2}$, then X is biholomorphically homeomorphic to $P_2(C)$.*

Theorem 2 (Andreotti-Frankel). *If X has positive sectional curvature, i.e. $K(p) > 0$ for all p, then X is biholomorphically homeomorphic to $P_2(C)$.*

Theorem 2′ (Goldberg-Kobayashi). *If X has positive holomorphic bisectional curvature, then X is biholomorphically homeomorphic to $P_2(C)$.*

Remarks. In Theorem 1, the number $\frac{1}{2}$ cannot be reduced, since the metric product $P_1(C) \times P_1(C)$ with the Fubini-Study metric on each factor has holomorphic pinching $\frac{1}{2}$. Theorem 2 is due to Andreotti and Frankel (Frankel [4]); their proof and the proof by Goldberg and Kobayashi of the closely related Theorem 2′ (Goldberg and Kobayashi [5]) as well depend upon the classification of algebraic surfaces. Our proofs are quite short and are independent of the classification theory.

Proof of Theorems 1, 2 and 2′. Since the proofs of the three theorems are very similar, we give the proofs together. Suppose that X satisfies the hypothesis of one of the Theorems 1, 2, or 2′. Then:

(a) The Ricci tensor of X is positive definite. This fact follows immediately under the hypothesis of positivity of sectional curvature and follows from curvature computations given in Berger [1] and Goldberg-Kobayashi [5] under the hypotheses of δ-holomorphic pinching, $\delta > \frac{1}{2}$, and positivity of holomorphic bisectional curvature, respectively.

(b) $H^2(X, R) = R$. This result, which is proved using a technique of Bochner and Lichnerowicz, is proved under the hypothesis of δ-holomorphic pinching, $\delta > \frac{1}{2}$, by Bishop and Goldberg [2] and under the hypothesis of positivity of holomorphic bisectional curvature by Goldberg and Kobayashi [5].

(c) X is simply connected. A theorem of Kobayashi (Kobayashi [10]) states that any compact Kähler manifold with positive definite Ricci tensor is simply connected; thus (c) follows from (a).

(d) The first Chern class $c_1(X)$ is a positive multiple of the Kähler class of X. This statement follows from (a) and (b) and the standard formula expressing the Chern classes in terms of the curvature form (Chern [3]).

Using these facts we can compute the first Chern and Pontrjagin classes of X. Since X is simply-connected, $H^2(X: Z)$ is free, by the universal coefficient theorem; since the second Betti number of X is 1, $H^2(X: Z)$ is infinite cyclic. Choose ξ to be the generator which is a *positive* multiple of the Kähler class ω. By Poincaré duality, ξ^2 is a generator for $H^4(X: Z)$. In fact it is the class dual to the orientation class, i.e., $\xi^2[X] = 1$. This is because ξ^2 is a positive multiple of $\omega \wedge \omega$, which is a positive multiple of the volume form. Thus the integral cohomology ring is easily seen to be $H^*(X: Z) = Z[\xi]/(\xi^3)$.[1]

Therefore the signature of X is 1, and by the Hirzebruch-Thom theorem [6] the first Pontrjagin class $p_1(X) = 3\xi^2$. Now we have the relation $c_1(X)^2 = 2c_2(X) + p_1(X)$. (This relation is valid quite generally for complex vector-bundles—see Hirzebruch [6, Chapter 1], or Hirzebruch and Hopf [7] for the case of 4-manifolds.) In the case at hand, $c_2(X)$ is the Euler class, so $c_1(X)^2 = 9\xi^2$. Since $c_1(X)$ is a positive multiple of ξ, $c_1(X) = 3\xi$.

One can now apply the argument of Hirzebruch and Kodaira [8] as given for example in Morrow [13]. Morrow shows that if X is homeomorphic to $P_2(C)$ and the first Chern class is positive, then X is necessarily biholomorphically homeomorphic to $P_2(C)$. However, the assumption that X is homeomorphic to $P_2(C)$ is actually used only to calculate the $\hat{A}$-genus of X and thence the Chern class $c_1(X)$. Since we already know $c_1(X)$ and $p_1(X)$ we can continue the reasoning without the homeomorphism assumption: Since ξ is of the type (1, 1), there is a holomorphic complex line bundle $E \to X$ corresponding to ξ. By the Riemann-Roch theorem and the Kodaira vanishing theorem, one finds that $\dim_C \Gamma(E) = \dim_C H^0(X, O(E)) = 3$ and that, for some large integer s, E^s is ample. Then the "meromorphic" map ϕ: $X \to P_2(C)$ given by three linearly independent sections $\{\phi_0, \phi_1, \phi_2\}$ of E can be shown by a standard argument using the Segre embedding to be a biholomorphic homeomorphism (Morrow [13, pp. 319–320]; cf. Howard [9], Kobayashi and Ochiai [12]).

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[1] This already implies that X has the homotopy type of CP_2, since $\pi_1(X) = 0$. However, we do not need this.

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