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RANDOM SHIFTS WHICH PRESERVE MEASURE[1]

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ABSTRACT. Given a flow θ_g, $g \in G$ a group, over a probability space $(\Omega, \mathcal{F}, P)$ and a G-valued random variable Z, we exhibit the Lebesgue decomposition of the measure $P \circ \theta_Z^{-1}$ relative to P, and give necessary and sufficient conditions for equality ($P \circ \theta_Z^{-1} = P$), absolute continuity ($P \circ \theta_Z^{-1} \ll P$), and singularity ($P \circ \theta_Z^{-1} \perp P$) in terms of the Haar measure. The proof rests on the theory of "Palm measures" as developed by Mecke and the authors. Specializing the group G, we retrieve some known results for the integers and real line, and compute the Radon-Nikodým derivatives in various cases.

0. **Introduction.** Let G be a locally compact, second countable, Abelian group with Borel σ-field $\mathcal{G}$ and Haar measure $\mu(dg)$ (or just dg). A *flow* $\theta = (\theta_g)$, $g \in G$, on a probability space $(\Omega, \mathcal{F}, P)$ is a group of bimeasurable, measure-preserving bijections $\theta_g\colon \Omega \rightarrow \Omega$ such that $\theta_0 =$ identity and the mapping $(g, \omega) \rightarrow \theta_g(\omega)$ is measurable. If $Z\colon \Omega \rightarrow G$ is measurable, we write θ_Z for the measurable function $\omega \rightarrow \theta_{Z(\omega)}(\omega)$, $\psi_Z(g, \omega) = g + Z \circ \theta_g(\omega)$, and $P_Z = P \circ \theta_Z^{-1}$.

A natural question in the study of flows is to characterize those random variables $Z\colon \Omega \rightarrow G$ for which θ_Z is a measure-preserving transformation, i.e. $P_Z = P$. Neveu [8] gave the solution for the case $G = \mathbf{Z}$ (the integers), and Dinges [1] obtained some special results for $G = \mathbf{R}$ (the real line). Finally, it can be deduced from [6, Satz 4.3] that a sufficient condition for $P_Z = P$ is that $\psi_Z(\cdot, \omega)$ be a Haar-measure preserving transformation on G for almost every $\omega \in \Omega$.

Here we exhibit the Lebesgue decomposition of P_Z relative to P using the theory of "Palm measures" [4], [5]. We can then show that the above condition is also necessary for $P_Z = P$, and give necessary and sufficient conditions for P_Z to be absolutely continuous relative to P (written $P_Z \ll P$) or equivalent to P (i.e. have the same sets of measure zero). Finally,

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by making appropriate choices for the group G, we retrieve and expand some of the results of Neveu and Dinges mentioned above.

§1 contains some preliminary work on Palm measures, the main result is given in §2, and §3 consists of examples.

If $\mathfrak{F}$ is a σ-field, we shall use the (ambiguous) notation $\xi \in (\mathfrak{F})$ to mean that the function ξ is measurable relative to $\mathfrak{F}$, the range being clear from context; $\xi \in (\mathfrak{F})_+$ indicates the range is $[0, \infty)$.

1. **Palm measures.** A *homogeneous random measure* (HRM) $\alpha(B, \omega)$ is a function on $\mathcal{G} \times \Omega$ which is (i) a Radon measure on $\mathcal{G}$ a.s., (ii) a random variable on Ω for all $B \in \mathcal{G}$, and (iii) homogeneous relative to θ in that $\alpha(B + g, \omega) = \alpha(B, \theta_g\omega)$ $\forall g \in G$, $B \in \mathcal{G}$ a.s., where $B + g$ is the translate of B by g. (When $G = \mathbf{R}$, $\alpha((0, t], \omega)$ defines an *additive functional.*)

Note. Let N and M denote the exceptional ω-sets in (i) and (iii) respectively. Clearly M is θ-invariant; in fact, so is $N \cup M$, so that we can and do assume (i) and (iii) hold off a P-null invariant set.

The Palm measure of α is defined by

$$\hat{P}_\alpha(A) = E \int_G \zeta(g) I_A \circ \theta_g(\omega)\alpha(dg, \omega), \qquad A \in \mathfrak{F},$$

where $\zeta: G \to \mathbf{R}$ is such that $\int_G \zeta(g)\, dg = 1$. I_A is the indicator of the set A. The measure $\hat{P}_\alpha$ is independent of the choice of ζ (cf. the argument in [5]), and is always σ-finite. Further, $\hat{P}_\alpha$ is finite iff $E\alpha(B) < \infty$ for B compact, in which case $E\alpha(dg) = \text{const} \times \mu(dg)$; α is then called *integrable.* A monotone class argument gives

$$\hat{E}_\alpha(\xi) = E \int_G \zeta(g)\xi \circ \theta_g(\omega)\alpha(dg, \omega), \qquad \xi \in (\mathfrak{F})_+, \tag{1}$$

$\hat{E}_\alpha$ being integration with $\hat{P}_\alpha$.

Proceeding exactly as in the case $G = \mathbf{R}$ [4] one obtains

$$E \int_G Y(g, \theta_g\omega)\alpha(dg, \omega) = \hat{E}_\alpha \int_G Y(g, \omega)\, dg, \qquad Y \in (\mathcal{G} \otimes \mathfrak{F})_+. \tag{2}$$

Choosing $Y(g, \omega) = I_B(g)\xi \circ \theta_{-g}(\omega)$, we have $E(\xi\alpha(B)) = \hat{E}_\alpha \int_B \xi \circ \theta_{-g}\, dg$, $B \in \mathcal{G}$, $\xi \in (\mathfrak{F})_+$. Since $\mathcal{G}$ is separable, this leads to

(3) **Lemma.** *Let α, β be HRM's. Then $\hat{P}_\alpha = \hat{P}_\beta$ iff $\alpha(dg, \omega) = \beta(dg, \omega)$ for almost every $\omega \in \Omega$.*

From (1) we get

(4) **Lemma.** *Let Q be a σ-finite measure on $\mathfrak{F}$ such that $Q \ll \hat{P}_\alpha$ for*

some HRM α. *Then* $Q = \hat{P}_\beta$, *where* $\beta(dg, \omega) = \xi \circ \theta_g(\omega)\alpha(dg, \omega)$ *and* $\xi = dQ/d\hat{P}_\alpha$.

Before going on, let us observe that any HRM $\alpha(dg, \omega)$ is σ-finite on $\mathcal{G}$ a.s., hence, using [7, p. 154], it is not difficult to prove that there exists a function $\chi \in (\mathcal{G} \otimes \mathcal{F})_+$ which, for such ω, serves as the density of the μ-absolutely continuous part of α: $\alpha(dg, \omega) = \chi(g, \omega)dg + \gamma(dg, \omega)$, where $\gamma(dg, \omega)$ is a.s. μ-singular (written $\gamma(dg, \omega) \perp \mu(dg)$ a.s.).

(5) **Theorem.** *Let* $\beta(dg, \omega) = \chi(g, \omega)dg$ *be the* μ*-absolutely continuous part of* α, *and* $\gamma(dg, \omega)$ *the* μ*-singular part. Then* β *and* γ *are HRM's and* $\hat{P}_\alpha = \hat{P}_\beta + \hat{P}_\gamma$ *is Lebesgue decomposition of* $\hat{P}_\alpha$ *relative to* P: $\hat{P}_\beta \ll P$, $\hat{P}_\gamma \perp P$.

Proof. Using the homogeneity of α, if $h \in G$, $B \in \mathcal{G}$, and $\omega \notin N \cup M$, we have

$$\begin{aligned}\alpha(B + h, \omega) &= \int_{B+h} \chi(g, \omega)\, dg + \gamma(B + h, \omega)\\ &= \int_B \chi(g + h, \omega)\, dg + \gamma(B + h, \omega)\\ &= \int_B \chi(g, \theta_h\omega)\, dg + \gamma(B, \theta_h\omega).\end{aligned}$$

Now $\gamma(B + h, \omega)$ is μ-singular (as a measure in B), so, by uniqueness of Lebesgue decomposition, $\gamma(B + h, \omega) = \gamma(B, \theta_h\omega)$, i.e. γ is an HRM. Hence β is also an HRM, and, obviously, $\hat{P}_\alpha = \hat{P}_\beta + \hat{P}_\gamma$.

Let $\zeta: G \to \mathbf{R}$ have $\int_G \zeta(g)\, dg = 1$. For any $\xi \in (\mathcal{F})_+$, we then have

$$\hat{E}_\beta(\xi) = E \int_G \zeta(g)\chi(g, \omega)\xi \circ \theta_g(\omega)\, dg = E\xi \int_G \zeta(g)\chi(g, \theta_{-g}\omega)\, dg$$

which shows $\hat{P}_\beta \ll P$. Since ζ is arbitrary we note also that $\chi(g, \theta_{-g}(\omega))$ is a.s. constant in g, so we can take $\hat{E}_\beta(\xi) = E(\xi\chi)$, $\chi \in (\mathcal{F})_+$, and, by (3), $\beta(dg, \omega) = \chi \circ \theta_g(\omega)\, dg$ a.s.

Suppose, next, that $\hat{P}_\gamma$ is not P-singular. Write the Lebesgue decomposition $\hat{P}_\gamma = Q + Q'$, where $Q \ll P$ and $Q' \perp P$. By (4), both Q and Q' are Palm measures. Since P is the Palm measure of $\mu(dg)$, (4) tells us that Q corresponds to a (nontrivial) HRM of the form $\eta \circ \theta_g dg$. Let ρ be the HRM corresponding to Q'. Then, by (3), $\gamma(dg, \omega) = \eta \circ \theta_g(\omega)\, dg + \rho(dg, \omega)$ a.s., contradicting the a.s. μ-singularity of γ. Q.E.D.

(6) **Corollary.** *For an HRM* α, $\hat{P}_\alpha \ll P$ (*resp.* $\hat{P}_\alpha \perp P$) *iff* $\alpha(dg, \omega) \ll \mu(dg)$ *a.s.* (*resp.* $\alpha(dg, \omega) \perp \mu(dg)$ *a.s.*).

2. **Main result.** Let Z, ψ_Z, and P_Z be as in §0, and define a random measure by

(7) $$\alpha_Z(B, \omega) = \int_G I_B(\psi_Z(g, \omega))\, dg, \qquad B \in \mathcal{G}.$$

(8) **Lemma.** *The random measure* α_Z *is an HRM whose Palm measure is* P_Z.

Proof. First consider

$$E\alpha_Z(B) = E \int_G I_B(g + Z \circ \theta_g)\, dg$$
$$= E \int_G I_B(g + Z)\, dg \quad (\text{since } \theta_g \text{ is a flow})$$
$$= \mu(B).$$

Using the σ-compactness of G, we find that $\alpha_Z(B, \omega)$ is finite on compacts a.s. Next, to prove homogeneity, fix $h \in G$. We have

$$\alpha_Z(B + h, \omega) = \int_G I_{B+h}(g + Z \circ \theta_g(\omega))\, dg \;= \int_G I_B(g - h + Z \circ \theta_g(\omega))\, dg$$
$$= \int_G I_B(g + Z \circ \theta_{g+h}(\omega))\, dg \quad (\text{since } \theta_{g+h} = \theta_g \circ \theta_h)$$
$$= \alpha_Z(B, \theta_h\omega).$$

Finally, it is clear from (7) that, for any function $\phi \in (\mathcal{G})_+$,

(9) $$\int_G \phi(g)\alpha_Z(dg, \omega) = \int_G \phi(\psi_Z(g, \omega))\, dg.$$

It follows that, if $\xi \in (\mathcal{F})_+$ and $\zeta: G \to \mathbf{R}$ satisfies $\int_G \zeta(g)\, dg = 1$,

$$\hat{E}_{\alpha_Z}(\xi) = E \int_G \zeta(g + Z \circ \theta_g)\xi \circ \theta_{g + Z\circ\theta_g}\, dg$$
$$= E \int_G \zeta(g + Z \circ \theta_g)\xi \circ \theta_Z \circ \theta_g\, dg$$
$$= E \int_G \zeta(g + Z)\xi \circ \theta_Z\, dg = \int \xi\, dP_Z. \quad \text{Q.E.D.}$$

Note. Since $\alpha_Z(dg, \omega)$ is homogeneous $\forall \omega$ (i.e. $M = \emptyset$) and a measure $\forall \omega$, and since G has a countable basis of relatively compact sets, we need not complete $\mathcal{F}$ to ensure the measurability of the ω-set on which $\alpha_Z(dg, \omega)$ is finite on compacts.

As a consequence of (8), if $\beta(dg, \omega) + \gamma(dg, \omega)$ is the Lebesgue decomposition of $\alpha_Z(dg, \omega)$ relative to $\mu(dg)$, where $\beta(dg, \omega) \ll \mu(dg)$ and $\gamma(dg, \omega) \perp \mu(dg)$ a.s., we find the Lebesgue decomposition

$$P_Z = \hat{P}_\beta + \hat{P}_\gamma, \qquad \hat{P}_\beta \ll P, \qquad \hat{P}_\gamma \perp P.$$

Explicit expressions for $\hat{P}_\beta$ will be given in some special cases in §3.

Finally, using the results of §1, namely (3), (4), (6), and noting that $\alpha_Z(dg, \omega)$ is the measure $\mu \circ \psi_Z^{-1}(\cdot, \omega)$, we have

(10) **Theorem.** (a) $P_Z = P$ *iff* $\mu \circ \psi_Z^{-1}(\cdot, \omega) = \mu$ *a.s., i.e. iff the function* $\psi_Z(\cdot, \omega)$ *is* μ*-measure preserving a.s.*

(b) $P_Z \ll P$ *iff* $\mu \circ \psi_Z^{-1}(\cdot, \omega) \ll \mu$ *a.s.*

(c) P_Z *is equivalent to* P *iff* $\mu \circ \psi_Z^{-1}(\cdot, \omega)$ *is equivalent to* μ *a.s.*

3. **Examples.** 1. Let G be any countable group with the discrete topology, μ the counting measure. If α is any HRM and $\xi \in (\mathcal{F})_+$, we have from (1):

$$\hat{E}_\alpha(\xi) = E \sum_{k \in G} \zeta(k)\xi \circ \theta_k \alpha(\{k\})$$

$$= \sum_{k \in G} \zeta(k) E(\xi(\omega)\alpha(\{k\}, \theta_{-k}\omega)) = E(\xi\alpha(\{0\})),$$

where $\Sigma\zeta(k) = 1$. Hence $\hat{P}_\alpha \ll P$. In particular, $P_Z \ll P$ and the Radon-Nikodým derivative is

$$\alpha_Z(\{0\}) = \sum_{k \in G} I_{\{0\}}(k + Z \circ \theta_k) = \#\{k \in G\colon \psi_Z(k) = 0\},$$

where # denotes the cardinality ($\leq +\infty$) of the indicated set. For $P_Z = P$ it is then necessary and sufficient to have $1 = \Sigma_{k \in G} I_{\{k + Z \circ \theta_k = 0\}}$ a.s., a result due to Neveu [8] when $G = \mathbf{Z}$.

2. Let G be arbitrary, but suppose that Z is discrete. Computing P_Z directly, it happens again that $P_Z \ll P$ with derivative $\eta(\omega) = \#\{g \in G\colon \psi_Z(g, \omega) = 0\}$.

3. Consider the case $G = \mathbf{R}$ and assume $Z(\omega) \geq 0$ is a "terminal" random variable, meaning that $Z(\omega) = t + Z \circ \theta_t(\omega)$ whenever $Z(\omega) > t$, $t \in \mathbf{R}$. Consequently, $\alpha_Z(\{Z(\omega)\}, \omega) \geq Z(\omega)$. Thus, $\alpha(dt, \omega)$ has a singular component whenever $Z(\omega) > 0$, so that $P_Z \ll P$ is impossible unless $Z = 0$ a.s., in which case $P_Z = P$.

If $M(\omega)$ is a homogeneous random set, i.e. $M(\theta_t\omega) = M(\omega) - t$ for all t, ω, then $Z(\omega) = \inf\{t > 0\colon t \in M(\omega)\}$ is terminal, and an easy computation yields, for any $t > 0$,

$$\alpha((0, t], \omega) = \int_{-\infty}^{t} I_{(0,t]}(s + Z \circ \theta_s(\omega))\, ds = L(t^+, \omega) - L(0^+, \omega),$$

where $L(t, \omega) = \sup\{s \leq t: s \in M(\omega)\}$. By (8), the Palm measure of the additive functional $L(t^+) - L(0^+)$ is just $P \circ \theta_Z^{-1}$.

4. Consider again $G = \mathbf{R}$, and suppose now that, for almost every $\omega \in \Omega$, $Z \circ \theta_t(\omega)$ is an absolutely continuous function of $t \in \mathbf{R}$. By a Fubini argument it is easy to prove that there is a random variable Z' such that $Z' \circ \theta_t(\omega) = dZ \circ \theta_t(\omega)/dt$ for almost every t, a.s. The process $\psi_Z(t, \omega) = t + Z \circ \theta_t(\omega)$ is absolutely continuous and a.s. has derivative $\psi'(t, \omega) = 1 + Z' \circ \theta_t(\omega)$ a.e. To say that $\alpha_Z(\cdot, \omega) \ll \mu$ means that, if $\mu(B) = 0$ for $B \in \mathcal{B}$ (the Borel σ-field on $\mathbf{R}$), then the process $\psi(t, \omega) = \psi_Z(t, \omega)$ spends (Lebesgue measure) zero time in B, i.e. $\psi(t, \omega)$ has a local time relative to μ. Local times (or occupation time densities) for smooth processes were studied in [3], and we now apply some of those results to the process at hand. Let

$$\nu_x(U, \omega) = \#\{s \in U: \psi(s, \omega) = x\}, \qquad U \in \mathcal{B},$$

$$\beta(x, \omega) = \int_{-\infty}^{\infty} |\psi'(s, \omega)|^{-1} \nu_x(ds, \omega).$$

Now $\alpha_Z(B, \omega) = \int_{-\infty}^{\infty} I_B(\psi(t, \omega))\, dt$ is the total time spent by $\psi(\cdot, \omega)$ in B. From [3] we conclude that $\beta(x, \omega)$ is the density of the absolutely continuous component of the measure $\alpha_Z(\cdot, \omega)$. A necessary and sufficient condition for the absence of the singular component is that $\mu\{t: \psi'(t, \omega) = 0\} = 0$ a.s. By a Fubini argument and stationarity, this reduces to $P\{Z' = -1\} = 0$.

Consider the absolutely continuous component α_a of α_Z in more detail. We have

$$\beta(x, \omega) = \int_{-\infty}^{\infty} |1 + Z' \circ \theta_s|^{-1} \nu_x(ds, \omega) = \sum_{s: s + Z\circ\theta_s = x} |1 + Z' \circ \theta_s|^{-1}$$

$$= \sum_{s: \psi(s, \theta_x \omega) = 0} |1 + Z' \circ \theta_s \circ \theta_x(\omega)|^{-1} = \beta(0, \theta_x \omega).$$

Thus $\alpha_a(dt, \omega) = \beta(0, \theta_t \omega)\, dt$, so the Palm measure of α_a is $\hat{P}_{\alpha_a}(A) = E(\beta(0); A)$, $A \in \mathcal{F}$, i.e. the density of the absolutely continuous component of P_Z is $\beta(0) = \int_{-\infty}^{\infty} |1 + Z' \circ \theta_s|^{-1} \nu_0(ds)$. This appears to generalize some of the work of Dinges [1], though we have been unable to comprehend fully his results. It is also a continuous analogue of Example 2.

Finally we note that, if Z is integrable, $\beta(0) > 0$ a.s. and so α_a is a.s. equivalent to μ, i.e. $\hat{P}_{\alpha_a}$ is equivalent to P. Indeed, it may be shown that, a.s., the derivative $\psi'(s, \omega)$ is finite at each time s at which $\psi(s, \omega) = x$, for a.e. x. It remains only to show that $\psi(\cdot, \omega)$ hits every x, a.s. Suppose $\psi(s, \omega) < x$ for every s. Then $t^{-1} \int_0^t \psi(s, \omega)\, ds < x$ for all t. But $t^{-1} \int_0^t \psi(s, \omega)\, ds = t/2 + t^{-1} \int_0^t Z \circ \theta_s(\omega)\, ds$, and this converges, for almost every ω, to $\pm\infty$ as $t \to \pm\infty$.

5. We indicate here an extension of the result of Example 4 when $G = \mathbf{R}^n$, $n \geq 1$. Assume that $t \to Z \circ \theta_t(\omega)$ is a.s. a Lipschitz function of $t \in \mathbf{R}^n$. Then $\psi(t, \omega) = t + Z \circ \theta_t(\omega)$ is likewise, and Theorem 3.2.5 (see also 2.10.35) of [2] yields (omitting ω)

$$\int_U g(\psi(t)) J(t)\, dt = \int_{\mathbf{R}^n} g(y) \nu_y(U)\, dy \quad \text{a.s.} \tag{11}$$

for any Borel function g (Borel set U) on $\mathbf{R}^n$, where $J(t)$ is the absolute value of the Jacobian of $\psi(t)$, which exists a.e., and $\nu_y(U)$ is defined as in Example 4. Arguing as in §1 of [3] we find

$$\int_{\mathbf{R}^n} h(t) g(\psi(t)) J(t)\, dt = \int_{\mathbf{R}^n} g(y) \int_{\mathbf{R}^n} h(t) \nu_y(dt)\, dy \tag{12}$$

for any nonnegative Borel function $h(t)$. Let N be a Borel set which is almost equal to $\{t: J(t) = 0\}$ (which is, in general, only Lebesgue measurable). Then [2, Theorem 3.2.3] shows $\nu_y(N) = 0$ for a.e. y. Now take $h(t) = (J(t))^{-1} I_{N^c}(t)$ a.e. From (12) we have

$$\int_{\mathbf{R}^n} g(\psi(t)) I_{N^c}(t)\, dt = \int_{\mathbf{R}^n} g(y) \int_{\mathbf{R}^n} \frac{\nu_y(dt)}{J(t)}\, dy,$$

or, with $g = I_B$,

$$\int_{N^c} I_B(\psi(t))\, dt = \int_B \int_{\mathbf{R}^n} \frac{\nu_y(dt)}{J(t)}\, dy. \tag{13}$$

As in [3], one notes that the left member of (13) is the absolutely continuous component of α_Z, and we conclude: the P-absolutely continuous component of P_Z has density $\int_{\mathbf{R}^n} \nu_0(dt)/J(t)$.

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