ABSTRACT. Let $G$ be an infinite compact group with dual object $\Sigma$. Corresponding to each semigroup $T = \{T(\xi); \xi \geq 0\}$ of operators on $L_p(G)$, $1 \leq p < \infty$, which commutes with right translations, there is a semigroup $\mathcal{E} = \{E_\xi(\sigma); \xi \geq 0, \sigma \in \Sigma\}$ of $L_p(G)$ multipliers. If $T$ is strongly continuous, then $\{E_\xi(\sigma); \xi \geq 0\}$ is uniformly continuous for each $\sigma$. Conversely, a semigroup $\mathcal{E}$ of $L_p(G)$-multipliers determines a semigroup $T$ of operators on $L_p(G)$. $T$ is strongly continuous if each $E_\xi(\sigma)$ is uniformly continuous; and then there exist a function $A$ on $\Sigma$ and $\Sigma_0 \subset \Sigma$ such that $E_\xi(\sigma) = E_0(\sigma)\exp(\xi A\sigma)$ if $\sigma \in \Sigma_0$ and $E_\xi(\sigma) = 0$ if $\sigma \notin \Sigma_0$.

1. Introduction. Let $X$ be a Banach space and denote by $B(X)$ the Banach algebra of all bounded linear operators on $X$ with the operator norm. A family $T = \{T(\xi); \xi \geq 0\}$ of operators in $B(X)$ is called a strongly continuous semigroup of operators on $X$ if

(i) $T(\xi_1 + \xi_2)x = T(\xi_1)[T(\xi_2)x], \xi_1, \xi_2 \in [0, \infty), x \in X$;

(ii) $\lim_{\xi \to 0^+} T(\xi)x = T(0)x, x \in X$.

If (i) holds and (ii) is replaced by

(iii) $\lim_{\xi \to 0^+} \|T(\xi) - T(0)\| = 0$,

then $T$ is called a uniformly continuous semigroup of operators on $X$.

Let $G$ be an infinite compact group with dual object $\Sigma$. We denote by $\mathcal{E}(\Sigma)$ the set $PB_{\sigma \in \Sigma}(H_\sigma)$, where $H_\sigma$ is the representation space of the representation $U(\sigma)$ [1, 28.24]. Suppose that $\mathcal{U}$ and $\mathcal{B}$ are subsets of $\mathcal{E}(\Sigma)$. An element $E$ of $\mathcal{E}(\Sigma)$ is said to be an $(\mathcal{U}, \mathcal{B})$-multiplier if $EA \in \mathcal{B}$ for all $A \in \mathcal{U}$ [1, 35.1]. If $E$ is an $(\mathcal{U}, \mathcal{U})$-multiplier, we shall call $E$, simply, an $\mathcal{U}$-multiplier.

Throughout this paper, $G$ denotes an infinite, compact group with dual object $\Sigma$. Haar measure on $G$ is denoted by $\lambda$, and it will be assumed that $\lambda$ is normalized so that $\lambda(G) = 1$. For $1 \leq p < \infty$, $L_p(G)$ denotes the usual Lebesgue space formed with respect to $\lambda$. The set of Fourier transforms $\hat{f}$ of $f \in L_p(G)$ will be denoted by $\hat{L}_p(G)$. It is shown in [1, 28.34] that $\hat{L}_p(G)$ is a subset of $\mathcal{E}(\Sigma)$. To simplify our notation, we shall write $'L_p(G)$-multiplier' in place of $\hat{L}_p(G)$-multiplier'.

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By a *semigroup of $L_p(G)$-multipliers* we shall mean a function $E$ on $[0, \infty) \times \Sigma$ such that

(i) for each pair $(\xi, \sigma)$, $E_\xi(\sigma) \in B(H_\sigma)$;
(ii) for each fixed $\xi$, $E_\xi(\cdot)$ is an $L_p(G)$-multiplier;
(iii) for each fixed $\sigma$, $\{E_\xi(\sigma); \xi \geq 0\}$ is a semigroup of operators on $H_\sigma$.

The results of this paper can be summarized as follows. Given a semigroup $\mathcal{J} = \{T(\xi); \xi \geq 0\}$ of operators on $L_p(G)$, the elements of which commute with right translations, we associate with $\mathcal{J}$ a semigroup $\mathcal{E} = \{E_\xi(\sigma); \xi \geq 0, \sigma \in \Sigma\}$ of $L_p(G)$-multipliers. We show that if $\mathcal{J}$ is strongly continuous, then $\{E_\xi(\sigma); \xi \geq 0\}$ is uniformly continuous for each $\sigma$. Conversely, given a semigroup $\mathcal{E} = \{E_\xi(\sigma); \xi \geq 0, \sigma \in \Sigma\}$ of $L_p(G)$-multipliers, we associate with $\mathcal{E}$ a semigroup $\mathcal{J} = \{T(\xi); \xi \geq 0\}$ of operators on $L_p(G)$, the members of which commute with right translations. We prove, moreover, that if $\{E_\xi(\sigma); \xi \geq 0\}$ is uniformly continuous for each $\sigma$, then $\mathcal{J}$ is strongly continuous. Furthermore, we show that there exist $A = (A_\sigma)$ in $S(\Sigma)$ and a subset $\Sigma_0$ of $\Sigma$ such that $E_\xi(\sigma) = E_0(\sigma)\exp(\xi A_\sigma)$ if $\sigma \in \Sigma_0$ and $E_\xi(\sigma) = 0$ if $\sigma \notin \Sigma_0$. Finally, we prove that if $\mathcal{A}$ is the infinitesimal operator of $\mathcal{J}$, then $A$ is a $(D(\mathcal{A}), L_p(G))$-multiplier.

The results and proofs in the present paper generalize to arbitrary infinite compact groups those of Hille [2, Theorems 20.3.1 and 20.3.2] for the circle group and those obtained in [5] for compact Abelian groups. It will be clear, however, that the orientation here is somewhat different from that of [2] and [5]. Moreover, it is hoped that our proofs and results shed some light on the classical situation.

2. Preliminaries. Let $G$ and $\Sigma$ be as defined above. It will be assumed throughout this paper that, for each $\sigma \in \Sigma$, a fixed representation $U(\sigma)$ with representation space $H_\sigma$ has been chosen and that, in each $H_\sigma$, a fixed conjugation $D_\sigma$ has been chosen. It will be understood that all Fourier-Stieltjes transforms and Fourier transforms are defined in terms of these fixed $U(\sigma)$'s and $D_\sigma$'s. In this and other definitions and notation, we follow Hewitt and Ross [1] where any undefined terms concerning harmonic analysis, used in this paper, will be found. Similarly, the reader is referred to Hille and Phillips [2] for an account of the theory of semigroups of operators on a Banach space.

2.1. Lemma. Let $\sigma \in \Sigma$ and for $U(\sigma)$ in $\sigma$ with representation space $H_\sigma$
let $\mathcal{S}_\sigma(G)$ denote the set of all finite complex linear combinations of functions of the form $\chi \mapsto \langle U(\sigma) \psi, \eta \rangle$ as $\psi, \eta$ vary over $H_\sigma$. Then $\mathcal{S}(\sigma) = \{f \in \mathcal{S}_\sigma(G); f(0) = 0\} = B(H_\sigma)$. 

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This is [1, Theorem (28.39)(i)].

2.2. Lemma. Let $T$ be a bounded linear operator on $L_p(G)$ which commutes with right translations. Then there exists a unique $E \in \mathcal{S}(\Sigma)$ such that $(Tf)\hat{\sigma} = E(\sigma)\hat{f}(\sigma)$ for all $f \in L_p(G)$ and all $\sigma \in \Sigma$.

Proof. Since $T$ commutes with right translations a routine argument shows that $T(f \ast g) = (Tf) \ast g$ for all $f, g \in L_p(G)$ (see e.g. [1, p. 376]). The result now follows from Theorem 35.8 of [1] and Lemma 2.1 above.

3. Semigroups of operators and semigroups of multipliers.

3.1. Theorem. Let $\mathcal{T} = \{T(\xi); \xi \geq 0\}$ be a semigroup of bounded linear operators on $L_p(G)$, each of which commutes with right translations. Then $\mathcal{T}$ defines a semigroup $\mathcal{E} = \{E_\xi(\sigma); \xi \geq 0, \sigma \in \Sigma\}$ of $L_p(G)$-multipliers. If, in addition, $\mathcal{T}$ is strongly continuous, then, for each $\sigma \in \Sigma$, the set $\{E_\xi(\sigma); \xi \geq 0\}$ is a uniformly continuous semigroup of operators on $H_\sigma$.

Proof. By Lemma 2.2, there exists, for each $\xi \geq 0$, a unique $E_\xi \in \mathcal{S}(\Sigma)$ such that $(T(\xi)f)\hat{\sigma} = E_\xi \hat{f}$ for every $f \in L_p(G)$. To complete the proof of the first assertion of the theorem, we only need to show that for each fixed $\sigma \in \Sigma$, the set $\{E_\xi(\sigma); \xi \geq 0\}$ is a semigroup of operators on $H_\sigma$. We have, for $f \in L_p(G)$ and $\xi_1, \xi_2 \geq 0$,

$$E_{\xi_1+\xi_2}(\sigma)\hat{f}(\sigma) = (T(\xi_1+\xi_2)f)\hat{\sigma} = [T(\xi_1)(T(\xi_2)f)]\hat{\sigma} = E_{\xi_1}(\sigma)(T(\xi_2)f)\hat{\sigma} = E_{\xi_1}(\sigma)E_{\xi_2}(\sigma)\hat{f}(\sigma).$$

Since, by Lemma 2.1, there exists $f \in \mathcal{S}_\sigma(G)$ such that $\hat{f}(\sigma) = 1$, we have $E_{\xi_1+\xi_2}(\sigma) = E_{\xi_1}(\sigma)E_{\xi_2}(\sigma)$. Hence, $\mathcal{E} = \{E_\xi(\sigma); \xi \geq 0, \sigma \in \Sigma\}$ is a semigroup of $L_p(G)$-multipliers.

Suppose that $T(\xi)$ is strongly continuous and let $\sigma$ be a fixed element of $\Sigma$. By Lemma 2.1, there exists $t \in \mathcal{S}(G)$ such that $\hat{t}(\sigma) = \mathcal{S}_\sigma$. Let $\epsilon > 0$; there exists $\gamma > 0$ such that $|[T(\xi) - T(0)]t|_p < \epsilon$ for all $\xi$ satisfying $0 < \xi < \gamma$. We have

$$\|E_\xi(\sigma) - E_0(\sigma)\|_{B(H_\sigma)} = \|E_\xi(\sigma) - E_0(\sigma)\|_{\mathcal{S}} \quad [1, D.42]$$

$$= \|[(E_\xi(\sigma) - E_0(\sigma))\hat{t}(\sigma)]_{\mathcal{S}}\|_{\mathcal{S}} = \|[T(\xi) - T(0)]t\hat{\sigma}\|_{\mathcal{S}} \quad [1, 28.34]$$

$$\leq \|[T(\xi) - T(0)]t\|_p < \epsilon.$$
for all $\xi$ satisfying $0 < \xi < \gamma$. This concludes the proof.

3.2. **Corollary.** If $\mathcal{T}$ is strongly continuous, then there exist a subset $\Sigma_0$ of $\Sigma$ and an $A \in \mathfrak{C}(\Sigma)$ such that

$$E_{\xi}(\sigma) = \begin{cases} E_0(\sigma) \exp(\xi A_\sigma) & \text{if } \sigma \in \Sigma_0, \\ 0 & \text{if } \sigma \notin \Sigma_0. \end{cases}$$

**Proof.** By Theorem 9.6.1 of [2], $E_0(\sigma)$ is, for each $\sigma$, a projection operator and

$$E_{\xi}(\sigma) = E_{\xi}(\sigma)E_0(\sigma) = E_0(\sigma)E_{\xi}(\sigma).$$

In particular, if $E_0(\sigma) = 0$, then $E_{\xi}(\sigma) = 0$ for all $\xi$. If for a given $\sigma$, $E_0(\sigma)$ is not the zero operator, then there exists a (unique) $A_\sigma \in \mathcal{B}(H_\sigma)$ such that $E_{\xi}(\sigma) = E_0(\sigma)\exp(\xi A_\sigma)$. Now define $A \in \mathfrak{C}(\Sigma)$ by setting $A(\sigma) = A_\sigma$ for each $\sigma \in \Sigma$, and set $\Sigma_0 = \{ \sigma \in \Sigma : E_0(\sigma) \neq 0 \}$.

3.3. **Theorem.** Let $\mathcal{E} = \{E_{\xi}(\sigma); \xi \geq 0, \sigma \in \Sigma \}$ be a semigroup of $L_p(G)$-multipliers. Then $\mathcal{E}$ defines a semigroup $\mathcal{T} = \{T(\xi); \xi \geq 0 \}$ of bounded linear operators on $L_p(G)$, each of which commutes with right translations. If, in addition, for each $\sigma$, the set $\{E_{\xi}(\sigma); \xi \geq 0 \}$ is a uniformly continuous semigroup of operators on $H_\sigma$, then $\mathcal{T}$ is a strongly continuous semigroup of operators on $L_p(G)$.

**Proof.** For each $\xi \geq 0$, we define $T(\xi)$ on $L_p(G)$ by $(T(\xi)f)(\sigma) = E_{\xi} f$, $f \in L_p(G)$. Then, by [1, 35.2], $T(\xi)$ is a bounded linear operator on $L_p(G)$. That the operators $T(\xi)$ have the semigroup property follows directly from the definition. We show that $T(\xi)$ commutes with right translations. First, we note that if $f \in L_p(G)$, then, for each $x \in G$,

$$(1) \quad \hat{f}_x(\sigma) = \hat{f}(\sigma)\overline{U}^{(\sigma)}_{x^{-1}}$$

for each $\sigma \in \Sigma$. In fact, for all $\xi, \eta \in H_\sigma$,

$$\langle \hat{f}(\sigma)\overline{U}^{(\sigma)}_{x^{-1}} \xi, \eta \rangle = \langle \hat{f}(\sigma)(\overline{U}^{(\sigma)}_{x^{-1}} \xi), \eta \rangle$$

$$= \int_G \langle \overline{U}^{(\sigma)}_{y} \xi, \eta \rangle f(y) d\lambda(y) = \int_G \langle \overline{U}^{(\sigma)}_{yx^{-1}} \xi, \eta \rangle f(y) d\lambda(y)$$

$$= \int_G \langle \overline{U}^{(\sigma)}_{y} \xi, \eta \rangle f_x(y) d\lambda(y) = \langle \hat{f}_x(\sigma) \xi, \eta \rangle,$$

and hence $\hat{f}_x(\sigma) = \hat{f}(\sigma)U^{(\sigma)}_{x^{-1}}$, $\sigma \in \Sigma$.

Now $(T(\xi)f)_x(\sigma) = E_{\xi}(\sigma)\hat{f}_x(\sigma)$ and

$$([T(\xi)f]_x)^\wedge(\sigma) = (T(\xi)f)_x^\wedge(\sigma)\overline{U}^{(\sigma)}_{x^{-1}} \quad (by \ (1))$$

\begin{align*}
&= E_{\xi}(\sigma)\hat{f}(\sigma)\overline{U}^{(\sigma)}_{x^{-1}} = E_{\xi}(\sigma)\hat{f}_x(\sigma) \quad (again \ by \ (1)).
\end{align*}
We therefore have \((T(\xi)f_x)(\sigma) = (T(\xi)f_x)(\sigma)\) for all \(\sigma \in \Sigma\), which implies that 
\(T(\xi)f_x = (T(\xi)f_x)\) for each \(x \in G\).

Suppose now that for each \(\sigma \in \Sigma\), the set \(\{E_{\xi}(\sigma); \xi \geq 0\}\) is a uniformly continuous semigroup of operators on \(H_\sigma\). To show that \(\{T(\xi); \xi \geq 0\}\) is strongly continuous, we shall first show that for every coordinate function \(u\), \(||[T(\xi) - T(\xi_0)]u||_p \to 0\) as \(\xi \to \xi_0\). Let \(\sigma\) be an arbitrary, but fixed, element of \(\Sigma\). Let \(U^{(\sigma)} \in \sigma\) and let \(\{\xi_1; \xi_2; \ldots; \xi_d\} \) be a basis in \(H_\sigma\).

We consider the coordinate function \(u^{(\sigma)}_{jk}\) defined on \(G\) by \(u^{(\sigma)}_{jk}(x) = \langle U_x^{(\sigma)} \xi_k, \xi_j \rangle\), where \(j, k\) is a fixed pair from \(\{1, 2, \ldots, d\}\). We have, for all \(\sigma' \in \Sigma\),

\[
([T(\xi) - T(\xi_0)]u^{(\sigma)}_{jk})^{(\sigma')} = (E_{\xi} - E_{\xi_0})(\sigma')u^{(\sigma)}_{jk}(\sigma')
\]

\[
\begin{cases}
(E_{\xi} - E_{\xi_0})(\sigma')u^{(\sigma)}_{jk}(\sigma) & \text{if } \sigma' = \sigma, \\
0 & \text{if } \sigma' \neq \sigma,
\end{cases}
\]

by [1, p. 80, (2)]. Thus \(([T(\xi) - T(\xi_0)]u^{(\sigma)}_{jk}) \in S_{00}(\Sigma)\) and hence

\[
[T(\xi) - T(\xi_0)]u^{(\sigma)}_{jk}
\]

is a trigonometric polynomial [1, 28.39]. We now have

\[
||[T(\xi) - T(\xi_0)]u^{(\sigma)}_{jk}||_p \leq ||[T(\xi) - T(\xi_0)]u^{(\sigma)}_{jk}||_u
\]

\[
\leq ||[T(\xi) - T(\xi_0)]u^{(\sigma)}_{jk}||_{A(G)} \quad [1, 34.6]
\]

\[
= \sum_{\sigma' \in \Sigma} d_{\sigma'} ||([T(\xi) - T(\xi_0)]u^{(\sigma)}_{jk})^{(\sigma')}||_{\phi_1} \quad [1, 34.4]
\]

\[
= d_{\sigma} ||([T(\xi) - T(\xi_0)]u^{(\sigma)}_{jk})^{(\sigma)}||_{\phi_1}
\]

\[
= d_{\sigma} ||E_{\xi}(\sigma) - E_{\xi_0}(\sigma)||_{\phi_1} \cdot ||\hat{u}^{(\sigma)}_{jk}(\sigma)||_{\phi_1}
\]

\[
\leq d_{\sigma} ||E_{\xi}(\sigma) - E_{\xi_0}(\sigma)||_{\phi_\infty} \cdot ||\hat{u}^{(\sigma)}_{jk}(\sigma)||_{\phi_1} \quad [1, D.52]
\]

\[
= d_{\sigma} ||E_{\xi}(\sigma) - E_{\xi_0}(\sigma)||_{B(H_\sigma)} \cdot ||\hat{u}^{(\sigma)}_{jk}(\sigma)||_{\phi_1} \quad [1, D.42]
\]

\[
\to 0 \quad \text{as } \xi \to \xi_0,
\]

by the uniform continuity of \(E_{\xi}(\sigma)\). Hence, \(||[T(\xi) - T(\xi_0)]u||_p \to 0\) as \(\xi \to \xi_0\) for every coordinate function \(u\). By the linearity of the operators \(T(\xi)\),

\[
||[T(\xi) - T(\xi_0)]f||_p \to 0 \quad \text{as } \xi \to \xi_0
\]

for every function \(f \in \mathfrak{F}(G)\). That \(||[T(\xi) - T(\xi_0)]f||_p \to 0\) as \(\xi \to \xi_0\) for every \(f \in
\(L_p(G)\) now follows from the last assertion, the fact that \(\mathcal{F}(G)\) is dense in \(L_p(G)\) and the continuity of the operators \(T(\xi)\). This concludes the proof.

3.4. Let \(\mathcal{E}\) be as in Theorem 3.3 and suppose that, for each \(\sigma \in \Sigma\), the set \(\{E_\xi(\sigma); \xi \geq 0\}\) is a uniformly continuous semigroup of operators on \(H_\sigma\). Then there exist a subset \(\Sigma_0\) of \(\Sigma\) and an \(A \in \mathcal{E}(\Sigma)\) such that

\[
E_\xi(\sigma) = \begin{cases} E_0(\sigma) \exp(\xi A_\sigma) & \text{if } \sigma \in \Sigma_0, \\
0 & \text{if } \sigma \notin \Sigma_0. \end{cases}
\]

Let \(\mathcal{A}_0\) denote the infinitesimal operator of the semigroup \(\mathcal{T}\) generated by \(\mathcal{E}\). The following theorem gives some information about the relation between \(\mathcal{A}_0\) and \(A\). Here, as is usual, we set \(E_0(\sigma) = I_\sigma\).

3.5. Theorem. For each \(f\) in the domain \(D(\mathcal{A}_0)\) of \(\mathcal{A}_0\) and \(\sigma \notin \Sigma_0\), we have \(\hat{f}(\sigma) = 0\). Furthermore, \((\mathcal{A}_0 f)^\wedge = A\hat{f}\) for \(f \in D(\mathcal{A}_0)\); i.e., \(A\) is a \((D(\mathcal{A}_0), L_p(G))\)-multiplier. If, in particular, \(\mathcal{T}\) is of class (A) with infinitesimal generator \(\mathcal{A}\), then \(\Sigma_0 = \Sigma\). If, in addition, \(A \in \mathcal{E}_\infty(\Sigma)\), then \(D(\mathcal{A}_0) = \{f \in L_p(G) \cap L_p(G) : Af \in L_p(G)\}\), and \((\mathcal{A}_0 f)^\wedge = A\hat{f}\) for \(f \in D(\mathcal{A}_0)\), so that \(A\) is a \((D(\mathcal{A}_0), L_p(G))\)-multiplier.

Proof. Let \(\epsilon > 0\); then there exists \(\gamma > 0\) such that

\[
\|\mathcal{A}_0 f - [T(\eta)f - f]/\eta\|_p < \epsilon
\]

for \(0 < \eta < \gamma\). This implies that if \(\sigma \notin \Sigma_0\), then \(\hat{f}(\sigma) = 0\), and if \(\sigma \in \Sigma_0\),

\[
(\mathcal{A}_0 f)^\wedge(\sigma) = A\sigma\hat{f}(\sigma), \quad f \in D(\mathcal{A}_0).
\]

Let \(\mathcal{T}\) be of class (A). Then \(D(\mathcal{A}_0) \subset D(\mathcal{T})\) is dense in \(L_p(G)\). Suppose there exists \(\sigma_0 \in \Sigma\) such that \(\sigma_0 \notin \Sigma_0\) and choose \(f \in L_p(G)\) such that \(\hat{f}(\sigma_0) \neq 0\). Given \(\epsilon > 0\), there exists \(f' \in D(\mathcal{A}_0)\) such that \(\|f - f'\|_p < \epsilon\). Then

\[
\|\hat{f}(\sigma_0) - \hat{f}(\sigma_0)\|_{B(H_\sigma)} \leq \|f' - f\|_p < \epsilon,
\]

which, by the first part of the theorem, implies that \(f(\sigma_0) = 0\) a contradiction. This proves that \(\Sigma_0 = \Sigma\).

To prove the last assertion of the theorem, let \(\omega_0\) be the type of the semigroup \(\mathcal{T}\) and set \(\mathcal{L}_0 = \mathcal{U}T(\xi)[L_p(G); \xi \geq 0]\). For \(\lambda\) with \(\text{Re}(\lambda) > \omega_0\), let \(R(\lambda; \mathcal{T})\) denote the resolvent of \(\mathcal{T}\). Then [2, p. 342] there exists \(\omega_1 > \omega_0\) such that

\[
R(\lambda; \mathcal{T})f = \int_0^\infty e^{-\lambda \xi} T(\xi)f d\xi, \quad f \in \mathcal{L}_0, \text{ Re}(\lambda) > \omega_1.
\]

For each \(\sigma \in \Sigma\), write \(S_\sigma(f) = f(\sigma), f \in L_p(G)\). Then \(S_\sigma\) is a bounded linear
transformation on $L_p(G)$ into $B(H_\sigma)$, and for all $f \in \mathcal{S}_0$,

$$S_\sigma(R(\lambda; \mathcal{G})f) = \int_0^\infty e^{-\lambda \xi} S_\sigma(T(\xi)f) d\xi = \int_0^\infty e^{-\lambda \xi} E_\xi(\sigma)f(\sigma) d\xi$$

$$= \int_0^\infty e^{-\lambda \xi} e^{\xi A_\sigma - A_\sigma} f(\sigma) d\xi = \int_0^\infty \xi (A_\sigma - A_\sigma)^{-1} f(\sigma) d\xi,$$

for all $\lambda$ with $\text{Re}(\lambda) > \max(\omega_1, ||A||_\infty)$, [2, (11.2.3)]. Since $\mathcal{S}_0$ is dense in $L_p(G)$, [2, p. 342], we have

$$(R(\lambda; \mathcal{G})f)(\sigma) = (\lambda I_\sigma - A_\sigma)^{-1} f(\sigma)$$

for all $f \in L_p(G), \text{Re}(\lambda) > \max(\omega_1, ||A||_\infty)$. We now make use of the last assertion to prove that

$$D(\mathcal{G}) = \{ f \in L_p(G): A_\sigma f \in L_p(G) \}.$$

Let $f \in D(\mathcal{G})$ and let $\sigma$ be an arbitrary element of $\Sigma$. Choose $\lambda$ such that $\lambda > \max(\omega_1, ||A||_\infty)$. Then there exists $g \in L_p(G)$ such that $f = R(\lambda; \mathcal{G})g$, and we have

$$(\mathcal{G} f)(\sigma) = [\lambda R(\lambda, \mathcal{G}) g - g]^\sigma = \lambda (\lambda I_\sigma - A_\sigma)^{-1} g(\sigma) - g(\sigma)$$

$$= A_\sigma (\lambda I_\sigma - A_\sigma)^{-1} g(\sigma) = A_\sigma f(\sigma).$$

Since $\sigma$ was arbitrary, $(\mathcal{G} f)(\sigma) = A_\sigma f(\sigma)$ for every $\sigma \in \Sigma$. Thus, if $f \in D(\mathcal{G})$, then $A_\sigma f \in L_p(G)$. Conversely, suppose that $f$ is an element of $L_p(G)$ such that $A_\sigma f \in L_p(G)$. Thus, there exists $h \in L_p(G)$ such that $A_\sigma f(\sigma) = h(\sigma)$ for all $\sigma \in \Sigma$. The function $g = \lambda f - h \in L_p(G)$ for all complex numbers $\lambda$. If $\lambda > \max(\omega_1, ||A||_\infty)$, we have

$$(R(\lambda; \mathcal{G})g)(\sigma) = (\lambda I_\sigma - A_\sigma)^{-1} g(\sigma)$$

$$= (\lambda I_\sigma - A_\sigma)^{-1} (\lambda f(\sigma) - A_\sigma f(\sigma)) = f(\sigma),$$

for all $\sigma \in \Sigma$. Hence $R(\lambda; \mathcal{G})g = f$, which implies that $f \in D(\mathcal{G})$. This concludes the proof.

3.6. Remarks. As an example of the situation described in Theorem 3.1, we mention the heat-diffusion semigroup $\{ T^t; t \geq 0 \}$ of operators on $L_p(G)$ for a compact Lie group $G$ discussed by Stein [4, p. 38]. Also, one obtains an illustration of Theorem 3.3 by considering the Fourier-Stieltjes transforms of the semigroup $\{ \mu_t; t \geq 0 \}$ of measures in $M(G)$ studied by Hunt [3].

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