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SEMIGROUPS OF MULTIPLIERS ASSOCIATED WITH SEMIGROUPS OF OPERATORS

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ABSTRACT. Let G be an infinite compact group with dual object Σ. Corresponding to each semigroup $\mathfrak{T} = \{T(\xi);\ \xi \geq 0\}$ of operators on $L_p(G)$, $1 \leq p < \infty$, which commutes with right translations, there is a semigroup $\mathfrak{E} = \{E_\xi(\sigma);\ \xi \geq 0,\ \sigma \in \Sigma\}$ of $L_p(G)$ multipliers. If $\mathfrak{T}$ is strongly continuous, then $\{E_\xi(\sigma);\ \xi \geq 0\}$ is uniformly continuous for each σ. Conversely a semigroup $\mathfrak{E}$ of $L_p(G)$-multipliers determines a semigroup $\mathfrak{T}$ of operators on $L_p(G)$. $\mathfrak{T}$ is strongly continuous if each $E_\xi(\sigma)$ is uniformly continuous; and then there exist a function A on Σ and $\Sigma_0 \subset \Sigma$ such that $E_\xi(\sigma) = E_0(\sigma)\exp(\xi A_\sigma)$ if $\sigma \in \Sigma_0$ and $E_\xi(\sigma) = 0$ if $\sigma \notin \Sigma_0$.

1. **Introduction.** Let X be a Banach space and denote by $B(X)$ the Banach algebra of all bounded linear operators on X with the operator norm. A family $\mathfrak{T} = \{T(\xi);\ \xi \geq 0\}$ of operators in $B(X)$ is called a *strongly continuous semigroup of operators on* X if

(i) $T(\xi_1 + \xi_2)x = T(\xi_1)[T(\xi_2)x]$, $\xi_1, \xi_2 \in [0, \infty)$, $x \in X$;

(ii) $\lim_{\xi \to 0^+} T(\xi)x = T(0)x$, $x \in X$.

If (i) holds and (ii) is replaced by

(iii) $\lim_{\xi \to 0^+} \|T(\xi) - T(0)\| = 0$,

then $\mathfrak{T}$ is called a *uniformly continuous semigroup of operators on* X.

Let G be an infinite compact group with dual object Σ. We denote by $\mathfrak{E}(\Sigma)$ the set $PB_{\sigma\in\Sigma}(H_\sigma)$, where H_σ is the representation space of the representation $U^{(\sigma)}$ [1, 28.24]. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of $\mathfrak{E}(\Sigma)$. An element E of $\mathfrak{E}(\Sigma)$ is said to be an $(\mathfrak{A}, \mathfrak{B})$-*multiplier* if $EA \in \mathfrak{B}$ for all $A \in \mathfrak{A}$ [1, 35.1]. If E is an $(\mathfrak{A}, \mathfrak{A})$-multiplier, we shall call E, simply, an $\mathfrak{A}$-multiplier.

Throughout this paper, G denotes an infinite, compact group with dual object Σ. Haar measure on G is denoted by λ, and it will be assumed that λ is normalized so that $\lambda(G) = 1$. For $1 \leq p < \infty$, $L_p(G)$ denotes the usual Lebesgue space formed with respect to λ. The set of Fourier transforms $\hat{f}$ of $f \in L_p(G)$ will be denoted by $L_p(G)^\wedge$. It is shown in [1, 28.34] that $L_p(G)^\wedge$ is a subset of $\mathfrak{E}(\Sigma)$. To simplify our notation, we shall write '$L_p(G)$-multiplier' in place of '$L_p(G)^\wedge$-multiplier'.

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By a *semigroup of $L_p(G)$-multipliers* we shall mean a function E on $[0, \infty) \times \Sigma$ such that

(i) for each pair (ξ, σ), $E_\xi(\sigma) \in B(H_\sigma)$;

(ii) for each fixed ξ, $E_\xi(\cdot)$ is an $L_p(G)$-multiplier;

(iii) for each fixed σ, $\{E_\xi(\sigma); \xi \geq 0\}$ is a semigroup of operators on H_σ.

The results of this paper can be summarized as follows. Given a semigroup $\mathfrak{T} = \{T(\xi); \xi \geq 0\}$ of operators on $L_p(G)$, the elements of which commute with right translations, we associate with $\mathfrak{T}$ a semigroup $\mathfrak{E} = \{E_\xi(\sigma); \xi \geq 0, \sigma \in \Sigma\}$ of $L_p(G)$-multipliers. We show that if $\mathfrak{T}$ is strongly continuous, then $\{E_\xi(\sigma); \xi \geq 0\}$ is uniformly continuous for each σ. Conversely, given a semigroup $\mathfrak{E} = \{E_\xi(\sigma); \xi \geq 0, \sigma \in \Sigma\}$ of $L_p(G)$-multipliers, we associate with $\mathfrak{E}$ a semigroup $\mathfrak{T} = \{T(\xi); \xi \geq 0\}$ of operators on $L_p(G)$, the members of which commute with right translations. We prove, moreover, that if $\{E_\xi(\sigma); \xi \geq 0\}$ is uniformly continuous for each σ, then $\mathfrak{T}$ is strongly continuous. Furthermore, we show that there exist $A = (A_\sigma)$ in $\mathfrak{C}(\Sigma)$ and a subset Σ_0 of Σ such that $E_\xi(\sigma) = E_0(\sigma)\exp(\xi A_\sigma)$ if $\sigma \in \Sigma_0$ and $E_\xi(\sigma) = 0$ if $\sigma \notin \Sigma_0$. Finally, we prove that if $\mathfrak{A}$ is the infinitesimal operator of $\mathfrak{T}$, then A is a $(D(\mathfrak{A}), L_p(G))$-multiplier.

The results and proofs in the present paper generalize to arbitrary infinite compact groups those of Hille [2, Theorems 20.3.1 and 20.3.2] for the circle group and those obtained in [5] for compact Abelian groups. It will be clear, however, that the orientation here is somewhat different from that of [2] and [5]. Moreover, it is hoped that our proofs and results shed some light on the classical situation.

2. **Preliminaries.** Let G and Σ be as defined above. It will be assumed throughout this paper that, for each $\sigma \in \Sigma$, a fixed representation $U^{(\sigma)}$ with representation space H_σ has been chosen and that, in each H_σ, a fixed conjugation D_σ has been chosen. It will be understood that all Fourier-Stieltjes transforms and Fourier transforms are defined in terms of these fixed $U^{(\sigma)}$'s and D_σ's. In this and other definitions and notation, we follow Hewitt and Ross [1] where any undefined terms concerning harmonic analysis, used in this paper, will be found. Similarly, the reader is referred to Hille and Phillips [2] for an account of the theory of semigroups of operators on a Banach space.

2.1. **Lemma.** *Let $\sigma \in \Sigma$ and for $U^{(\sigma)}$ in σ with representation space H_σ let $\mathfrak{T}_\sigma(G)$ denote the set of all finite complex linear combinations of functions of the form $x \to \langle U_x^{(\sigma)}\xi, \eta\rangle$ as ξ, η vary over H_σ. Then $\{\hat{f}(\sigma): f \in \mathfrak{T}_\sigma(G)\} = B(H_\sigma)$.*

This is [1, Theorem (28.39)(i)].

2.2. **Lemma.** *Let T be a bounded linear operator on $L_p(G)$ which commutes with right translations. Then there exists a unique $E \in \mathfrak{E}(\Sigma)$ such that $(Tf)^{\wedge}(\sigma) = E(\sigma)\hat{f}(\sigma)$ for all $f \in L_p(G)$ and all $\sigma \in \Sigma$.*

Proof. Since T commutes with right translations a routine argument shows that $T(f * g) = (Tf) * g$ for all $f, g \in L_p(G)$ (see e.g. [1, p. 376]). The result now follows from Theorem 35.8 of [1] and Lemma 2.1 above.

3. **Semigroups of operators and semigroups of multipliers.**

3.1. **Theorem.** *Let $\mathfrak{T} = \{T(\xi);\ \xi \geq 0\}$ be a semigroup of bounded linear operators on $L_p(G)$, each of which commutes with right translations. Then $\mathfrak{T}$ defines a semigroup $\mathfrak{E} = \{E_\xi(\sigma);\ \xi \geq 0, \sigma \in \Sigma\}$ of $L_p(G)$-multipliers. If, in addition, $\mathfrak{T}$ is strongly continuous, then, for each $\sigma \in \Sigma$, the set $\{E_\xi(\sigma);\ \xi \geq 0\}$ is a uniformly continuous semigroup of operators on H_σ.*

Proof. By Lemma 2.2, there exists, for each $\xi \geq 0$, a unique $E_\xi \in \mathfrak{E}(\Sigma)$ such that $(T(\xi)f)^{\wedge} = E_\xi \hat{f}$ for every $f \in L_p(G)$. To complete the proof of the first assertion of the theorem, we only need to show that for each fixed $\sigma \in \Sigma$, the set $\{E_\xi(\sigma);\ \xi \geq 0\}$ is a semigroup of operators on H_σ. We have, for $f \in L_p(G)$ and $\xi_1, \xi_2 \geq 0$,

$$E_{\xi_1+\xi_2}(\sigma)\hat{f}(\sigma) = (T(\xi_1+\xi_2)f)^{\wedge}(\sigma) = [T(\xi_1)(T(\xi_2)f)]^{\wedge}(\sigma)$$
$$= E_{\xi_1}(\sigma)(T(\xi_2)f)^{\wedge}(\sigma) = E_{\xi_1}(\sigma)E_{\xi_2}(\sigma)\hat{f}(\sigma).$$

Since, by Lemma 2.1, there exists $f \in \mathfrak{T}_\sigma(G)$ such that $\hat{f}(\sigma) = I_\sigma$, we have $E_{\xi_1+\xi_2}(\sigma) = E_{\xi_1}(\sigma)E_{\xi_2}(\sigma)$. Hence, $\mathfrak{E} = \{E_\xi(\sigma);\ \xi \geq 0, \sigma \in \Sigma\}$ is a semigroup of $L_p(G)$-multipliers.

Suppose that $T(\xi)$ is strongly continuous and let σ be a fixed element of Σ. By Lemma 2.1, there exists $t \in \mathfrak{T}(G)$ such that $\hat{t}(\sigma) = \mathfrak{T}_\sigma$. Let $\epsilon > 0$; there exists $\gamma > 0$ such that $\|[T(\xi) - T(0)]t\|_p < \epsilon$ for all ξ satisfying $0 < \xi < \gamma$. We have

$$\|E_\xi(\sigma) - E_0(\sigma)\|_{B(H_\sigma)} = \|E_\xi(\sigma) - E_0(\sigma)\|_{\phi_\infty} \qquad [1, \text{D.42}]$$
$$= \|[E_\xi(\sigma) - E_0(\sigma)]\hat{t}(\sigma)\|_{\phi_\infty} = \|([T(\xi) - T(0)]t)^{\wedge}(\sigma)\|_{\phi_\infty}$$
$$\leq \|([T(\xi) - T(0)]t)^{\wedge}\|_\infty \qquad [1, 28.34]$$
$$\leq \|[T(\xi) - T(0)]t\|_1 \qquad [1, 28.36]$$
$$\leq \|[T(\xi) - T(0)]t\|_p < \epsilon$$

for all ξ satisfying $0 < \xi < \gamma$. This concludes the proof.

3.2. **Corollary.** *If $\mathfrak{T}$ is strongly continuous, then there exist a subset Σ_0 of Σ and an $A \in \mathfrak{C}(\Sigma)$ such that*

$$E_\xi(\sigma) = \begin{cases} E_0(\sigma) \exp(\xi A_\sigma) & \text{if } \sigma \in \Sigma_0, \\ 0 & \text{if } \sigma \notin \Sigma_0. \end{cases}$$

Proof. By Theorem 9.6.1 of [2], $E_0(\sigma)$ is, for each σ, a projection operator and

$$E_\xi(\sigma) = E_\xi(\sigma)E_0(\sigma) = E_0(\sigma)E_\xi(\sigma).$$

In particular, if $E_0(\sigma) = 0$, then $E_\xi(\sigma) = 0$ for all ξ. If for a given σ, $E_0(\sigma)$ is not the zero operator, then there exists a (unique) $A_\sigma \in B(H_\sigma)$ such that $E_\xi(\sigma) = E_0(\sigma)\exp(\xi A_\sigma)$. Now define $A \in \mathfrak{C}(\Sigma)$ by setting $A(\sigma) = A_\sigma$ for each $\sigma \in \Sigma$, and set $\Sigma_0 = [\sigma \in \Sigma\colon E_0(\sigma) \neq 0]$.

3.3. **Theorem.** *Let $\mathfrak{E} = \{E_\xi(\sigma);\ \xi \geq 0, \sigma \in \Sigma\}$ be a semigroup of $L_p(G)$-multipliers. Then $\mathfrak{E}$ defines a semigroup $\mathfrak{T} = \{T(\xi);\ \xi \geq 0\}$ of bounded linear operators on $L_p(G)$, each of which commutes with right translations. If, in addition, for each σ, the set $\{E_\xi(\sigma);\ \xi \geq 0\}$ is a uniformly continuous semigroup of operators on H_σ, then $\mathfrak{T}$ is a strongly continuous semigroup of operators on $L_p(G)$.*

Proof. For each $\xi \geq 0$, we define $T(\xi)$ on $L_p(G)$ by $(T(\xi)f)\hat{} = E_\xi \hat{f}$, $f \in L_p(G)$. Then, by [1, 35.2], $T(\xi)$ is a bounded linear operator on $L_p(G)$. That the operators $T(\xi)$ have the semigroup property follows directly from the definition. We show that $T(\xi)$ commutes with right translations. First, we note that if $f \in L_p(G)$, then, for each $x \in G$,

$$(1) \qquad \hat{f}_x(\sigma) = \hat{f}(\sigma)\overline{U}^{(\sigma)}_{x^{-1}}$$

for each $\sigma \in \Sigma$. In fact, for all $\xi, \eta \in H_\sigma$,

$$\begin{aligned} \langle \hat{f}(\sigma)\overline{U}^{(\sigma)}_{x^{-1}}\xi, \eta\rangle &= \langle \hat{f}(\sigma)(\overline{U}^{(\sigma)}_{x^{-1}}\xi), \eta\rangle \\ &= \int_G \langle \overline{U}^{(\sigma)}_y(\overline{U}^{(\sigma)}_{x^{-1}}\xi), \eta\rangle f(y)d\lambda(y) = \int_G \langle \overline{U}^{(\sigma)}_{yx^{-1}}\xi, \eta\rangle f(y)d\lambda(y) \\ &= \int_G \langle \overline{U}^{(\sigma)}_y\xi, \eta\rangle f_x(y)d\lambda(y) = \langle \hat{f}_x(\sigma)\xi, \eta\rangle, \end{aligned}$$

and hence $\hat{f}_x(\sigma) = \hat{f}(\sigma)U^{(\sigma)}_{x^{-1}}$, $\sigma \in \Sigma$.

Now $(T(\xi)f_x)\hat{}(\sigma) = E_\xi(\sigma)\hat{f}_x(\sigma)$ and

$$\begin{aligned} ([T(\xi)f]_x)\hat{}(\sigma) &= (T(\xi)f)\hat{}(\sigma)\overline{U}^{(\sigma)}_{x^{-1}} \quad \text{(by (1))} \\ &= E_\xi(\sigma)\hat{f}(\sigma)\overline{U}^{(\sigma)}_{x^{-1}} = E_\xi(\sigma)\hat{f}_x(\sigma) \quad \text{(again by (1))}. \end{aligned}$$

We therefore have $(T(\xi)f_x)\hat{}(\sigma) = ([T(\xi)f]_x)\hat{}(\sigma)$ for all $\sigma \in \Sigma$, which implies that $T(\xi)f_x = (T(\xi)f)_x$ for each $x \in G$.

Suppose now that for each $\sigma \in \Sigma$, the set $\{E_\xi(\sigma);\ \xi \geq 0\}$ is a uniformly continuous semigroup of operators on H_σ. To show that $\{T(\xi);\ \xi \geq 0\}$ is strongly continuous, we shall first show that for every coordinate function u, $\|[T(\xi) - T(\xi_0)]u\|_p \to 0$ as $\xi \to \xi_0$. Let σ be an arbitrary, but fixed, element of Σ. Let $U^{(\sigma)} \in \sigma$ and let $\{\xi_1, \xi_2, \cdots \xi_{d_\sigma}\}$ be a basis in H_σ. We consider the coordinate function $u_{jk}^{(\sigma)}$ defined on G by $u_{jk}^{(\sigma)}(x) = \langle U_x^{(\sigma)}\xi_k, \xi_j\rangle$, where j, k is a fixed pair from $\{1, 2, \cdots, d_\sigma\}$. We have, for all $\sigma' \in \Sigma$,

$$([T(\xi) - T(\xi_0)]u_{jk}^{(\sigma)})\hat{}(\sigma') = (E_\xi - E_{\xi_0})(\sigma')\hat{u}_{jk}^{(\sigma)}(\sigma')$$

$$= \begin{cases} (E_\xi - E_{\xi_0})(\sigma)u_{jk}^{(\sigma)}(\sigma) & \text{if } \sigma' = \sigma, \\ 0 & \text{if } \sigma' \neq \sigma, \end{cases}$$

by [1, p. 80, (2)]. Thus $([T(\xi) - T(\xi_0)]u_{jk}^{(\sigma)})\hat{} \in \mathfrak{C}_{00}(\Sigma)$ and hence

$$[T(\xi) - T(\xi_0)]u_{jk}^{(\sigma)}$$

is a trigonometric polynomial [1, 28.39]. We now have

$$\begin{aligned}
\|[T(\xi) - T(\xi_0)]u_{jk}^{(\sigma)}\|_p &\leq \|[T(\xi) - T(\xi_0)]u_{jk}^{(\sigma)}\|_u \\
&\leq \|[T(\xi) - T(\xi_0)]u_{jk}^{(\sigma)}\|_{A(G)} \qquad [1, 34.6] \\
&= \sum_{\sigma' \in \Sigma} d_{\sigma'}\|([T(\xi) - T(\xi_0)]u_{jk}^{(\sigma)})\hat{}(\sigma')\|_{\phi_1} \qquad [1, 34.4] \\
&= d_\sigma\|([T(\xi) - T(\xi_0)]u_{jk}^{(\sigma)})\hat{}(\sigma)\|_{\phi_1} \\
&= d_\sigma\|[E_\xi(\sigma) - E_{\xi_0}(\sigma)]\hat{u}_{jk}^{(\sigma)}(\sigma)\|_{\phi_1} \\
&\leq d_\sigma\|E_\xi(\sigma) - E_{\xi_0}(\sigma)\|_{\phi_\infty} \cdot \|\hat{u}_{jk}(\sigma)\|_{\phi_1} \qquad [1, D.52] \\
&= d_\sigma\|E_\xi(\sigma) - E_{\xi_0}(\sigma)\|_{B(H_\sigma)} \cdot \|\hat{u}_{jk}^{(\sigma)}(\sigma)\|_{\phi_1} \qquad [1, D.42] \\
&\to 0 \quad \text{as } \xi \to \xi_0,
\end{aligned}$$

by the uniform continuity of $E_\xi(\sigma)$. Hence, $\|[T(\xi) - T(\xi_0)]u\|_p \to 0$ as $\xi \to \xi_0$ for every coordinate function u. By the linearity of the operators $T(\xi)$,

$$\|[T(\xi) - T(\xi_0)]t\|_p \to 0 \quad \text{as } \xi \to \xi_0$$

for every function $t \in \mathfrak{T}(G)$. That $\|[T(\xi) - T(\xi_0)]f\|_p \to 0$ as $\xi \to \xi_0$ for every $f \in$

$L_p(G)$ now follows from the last assertion, the fact that $\mathfrak{S}(G)$ is dense in $L_p(G)$ and the continuity of the operators $T(\xi)$. This concludes the proof.

3.4. Let $\mathfrak{E}$ be as in Theorem 3.3 and suppose that, for each $\sigma \in \Sigma$, the set $\{E_\xi(\sigma);\ \xi \geq 0\}$ is a uniformly continuous semigroup of operators on H_σ. Then there exist a subset Σ_0 of Σ and an $A \in \mathfrak{C}(\Sigma)$ such that

$$E_\xi(\sigma) = \begin{cases} E_0(\sigma)\exp(\xi A_\sigma) & \text{if } \sigma \in \Sigma_0, \\ 0 & \text{if } \sigma \notin \Sigma_0. \end{cases}$$

Let $\mathfrak{A}_0$ denote the infinitesimal operator of the semigroup $\mathfrak{T}$ generated by $\mathfrak{E}$. The following theorem gives some information about the relation between $\mathfrak{A}_0$ and A. Here, as is usual, we set $E_0(\sigma) = I_\sigma$.

3.5. **Theorem.** *For each f in the domain $D(\mathfrak{A}_0)$ of $\mathfrak{A}_0$ and $\sigma \notin \Sigma_0$, we have $\hat{f}(\sigma) = 0$. Furthermore, $(\mathfrak{A}_0 f)\hat{} = A\hat{f}$ for $f \in D(\mathfrak{A}_0)$; i.e. A is a $(D(\mathfrak{A}_0), L_p(G))$-multiplier. If, in particular, $\mathfrak{T}$ is of class* (A) *with infinitesimal generator $\mathfrak{A}$, then $\Sigma_0 = \Sigma$. If, in addition, $A \in \mathfrak{C}_\infty(\Sigma)$, then*

$$D(\mathfrak{A}) = [f \in L_p(G) : A\hat{f} \in L_p(G)\hat{}\,],$$

and $(\mathfrak{A}f)\hat{} = A\hat{f}$ for $f \in D(\mathfrak{A})$, so that A is a $(D(\mathfrak{A}), L_p(G))$-multiplier.

Proof. Let $\epsilon > 0$; then there exists $\gamma > 0$ such that

$$\|\mathfrak{A}_0 f - [T(\eta)f - f]/\eta\|_p < \epsilon$$

for $0 < \eta < \gamma$. This implies that if $\sigma \notin \Sigma_0$, then $\hat{f}(\sigma) = 0$, and if $\sigma \in \Sigma_0$,

$$(\mathfrak{A}_0 f)\hat{}(\sigma) = A_\sigma \hat{f}(\sigma), \quad f \in D(\mathfrak{A}_0).$$

Let $\mathfrak{T}$ be of class (A). Then $D(\mathfrak{A}_0) \subset D(\mathfrak{A})$ is dense in $L_p(G)$. Suppose there exists $\sigma_0 \in \Sigma$ such that $\sigma_0 \notin \Sigma_0$ and choose $f \in L_p(G)$ such that $\hat{f}(\sigma_0) \neq 0$. Given $\epsilon > 0$, there exists $f' \in D(\mathfrak{A}_0)$ such that $\|f - f'\|_p < \epsilon$. Then

$$\|\hat{f}'(\sigma_0) - \hat{f}(\sigma_0)\|_{B(H_{\sigma_0})} \leq \|f' - f\|_p < \epsilon,$$

which, by the first part of the theorem, implies that $f(\sigma_0) = 0$ a contradiction. This proves that $\Sigma_0 = \Sigma$.

To prove the last assertion of the theorem, let ω_0 be the type of the semigroup $\mathfrak{T}$ and set $\mathfrak{L}_0 = \bigcup\{T(\xi)[L_p(G)];\ \xi \geq 0\}$. For λ with $\mathrm{Re}(\lambda) > \omega_0$, let $R(\lambda; \mathfrak{A})$ denote the resolvent of $\mathfrak{A}$. Then [2, p. 342] there exists $\omega_1 > \omega_0$ such that

$$R(\lambda; \mathfrak{A})f = \int_0^\infty e^{-\lambda\xi}\, T(\xi) f\, d\xi, \qquad f \in \mathfrak{L}_0,\ \mathrm{Re}(\lambda) > \omega_1.$$

For each $\sigma \in \Sigma$, write $S_\sigma(f) = \hat{f}(\sigma)$, $f \in L_p(G)$. Then S_σ is a bounded linear transformation on $L_p(G)$ into $B(H_\sigma)$, and for all $f \in \mathfrak{L}_0$,

$$S_\sigma(R(\lambda; \mathfrak{A})f) = \int_0^\infty e^{-\lambda\xi} S_\sigma(T(\xi)f)d\xi = \int_0^\infty e^{-\lambda\xi} E_\xi(\sigma)\hat{f}(\sigma)d\xi$$

$$= \int_0^\infty e^{-\lambda I_\sigma \xi} e^{\xi A_\sigma}\hat{f}(\sigma)d\xi = \int_0^\infty e^{\xi(A_\sigma - \lambda I_\sigma)} d\xi\hat{f}(\sigma) = (\lambda I_\sigma - A_\sigma)^{-1}\hat{f}(\sigma),$$

for all λ with $\mathrm{Re}(\lambda) > \max(\omega_1, \|A\|_\infty)$, [2, (11.2.3)]. Since $\mathfrak{L}_0$ is dense in $L_p(G)$, [2, p. 342], we have

$$(R(\lambda; \mathfrak{A})f)\hat{}(\sigma) = (\lambda I_\sigma - A_\sigma)^{-1}\hat{f}(\sigma)$$

for all $f \in L_p(G)$, $\mathrm{Re}(\lambda) > \max(\omega_1, \|A\|_\infty)$. We now make use of the last assertion to prove that

$$D(\mathfrak{A}) = [f \in L_p(G): A\hat{f} \in L_p(G)\hat{}].$$

Let $f \in D(\mathfrak{A})$ and let σ be an arbitrary element of Σ. Choose λ such that $\lambda > \max(\omega_1, \|A\|_\infty)$. Then there exists $g \in L_p(G)$ such that $f = R(\lambda; \mathfrak{A})g$, and we have

$$(\mathfrak{A}f)\hat{}(\sigma) = [\lambda R(\lambda, \mathfrak{A})g - g]\hat{}(\sigma) = \lambda(\lambda I_\sigma - A_\sigma)^{-1}\hat{g}(\sigma) - \hat{g}(\sigma)$$

$$= A_\sigma(\lambda I_\sigma - A_\sigma)^{-1}\hat{g}(\sigma) = A_\sigma\hat{f}(\sigma).$$

Since σ was arbitrary, $(\mathfrak{A}f)\hat{}(\sigma) = A_\sigma\hat{f}(\sigma)$ for every $\sigma \in \Sigma$. Thus, if $f \in D(\mathfrak{A})$, then $A\hat{f} \in L_p(G)\hat{}$. Conversely, suppose that f is an element of $L_p(G)$ such that $A\hat{f} \in L_p(G)\hat{}$. Thus, there exists $h \in L_p(G)$ such that $A_\sigma\hat{f}(\sigma) = \hat{h}(\sigma)$ for all $\sigma \in \Sigma$. The function $g = \lambda f - h \in L_p(G)$ for all complex numbers λ. If $\lambda > \max(\omega_1, \|A\|_\infty)$, we have

$$(R(\lambda; \mathfrak{A})g)\hat{}(\sigma) = (\lambda I_\sigma - A_\sigma)^{-1}\hat{g}(\sigma)$$

$$= (\lambda I_\sigma - A_\sigma)^{-1}(\lambda\hat{f}(\sigma) - A_\sigma\hat{f}(\sigma)) = \hat{f}(\sigma),$$

for all $\sigma \in \Sigma$. Hence $R(\lambda; \mathfrak{A})g = f$, which implies that $f \in D(\mathfrak{A})$. This concludes the proof.

3.6. **Remarks.** As an example of the situation described in Theorem 3.1, we mention the heat-diffusion semigroup $\{T^t;\ t \geq 0\}$ of operators on $L_p(G)$ for a compact Lie group G discussed by Stein [4, p. 38]. Also, one obtains an illustration of Theorem 3.3 by considering the Fourier-Stieltjes transforms of the semigroup $\{\mu_t;\ t \geq 0\}$ of measures in $M(G)$ studied by Hunt [3].

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