AN ERGODIC SUPER-PROPERTY OF BANACH SPACES
DEFINED BY A CLASS OF MATRICES

A. BRUNEL, H. FONG1 AND L. SUCHESTON1

ABSTRACT. A matrix \( (a_{ni}) \) is called an \( R \)-matrix if (A) \( \sum_i a_{ni} \not\to 0 \), and (B) \( \lim_n a_{ni} = 0 \) for each \( i \). A Banach space \( X \) is called \( R \)-ergodic if for each isometry \( T \) and each \( x \in X \), there is an \( R \)-matrix \( (a_{ni}) \) such that \( \sum_i a_{ni} T^i x \rightharpoonup x \) (converges weakly). Given two Banach spaces \( F \) and \( X \), write \( F \text{fr } X \) if for each finite-dimensional subspace \( F' \) of \( F \) and \( \varepsilon > 0 \), there is an isomorphism \( V \) from \( F' \) onto a subspace of \( X \) such that 
\[
\|x - Vx\| < \varepsilon \quad \text{for each } x \in F' \text{ with } \|x\| \leq 1.
\]
\( X \) is called super-\( R \)-ergodic if \( F \text{fr } X \) for each \( F \text{fr } X \).

Theorem. \( X \) is super-\( R \)-ergodic if and only if \( X \) is super-reflexive.

The proof is based on the following:

Theorem. Let \( T \) be a linear operator on \( X \), \( (a_{ni}) \) a matrix satisfying (A), \( x \in X \) such that \( \sum_i a_{ni} T^i x \rightharpoonup \alpha x \). Then there is a constant \( a \) such that \( \|x - ax\| \in (I - T)X \).

A matrix \( (a_{ni}) \) with real terms is called an \( R \)-matrix iff it satisfies the following conditions:

(A) \[
\sum_i a_{ni} \not\to 0 \quad \text{as } n \to \infty;
\]

(B) \[
\lim_n a_{ni} = 0 \quad \text{for each } i.
\]

Condition (A) means that \( \sum_i a_{ni} \) exists for each \( n \) and the sequence \( (\sum_i a_{ni}) \) either diverges or converges to a limit different from zero.

A Banach space \( X \) is called \( R \)-ergodic iff for each isometry \( T \) and each \( x \in X \) there exists an \( R \)-matrix \( (a_{ni}) \) such that \( \sum_i a_{ni} T^i x \) converges weakly.

It is shown that \( X \) is super-\( R \)-ergodic if and only if it is super-stable (equivalently, super-reflexive). Since \( R \)-ergodicity is clearly implied by

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ergodicity, this is an improvement over the results of [1] and [2], to which the present theorem is reduced by an ergodic argument. It is further observed that a Banach space $X$ is reflexive if and only if for each bounded sequence $(x_i)$ in $X$ there is an $R$-matrix $(a_{ni})$ such that $\sum_i a_{ni} x_i \xrightarrow{w} (\xrightarrow{w}$ means: converges weakly).

1. A real Banach space $X$ is given, with elements $x, y, \cdots$. Sequences of real numbers are denoted by $a = (a_i), b = (b_i)$, etc. $S$ is the set of all sequences $(a_i)$ such that $a_i \neq 0$ for only finitely many indices $i$. Whenever we write $\sum a_i x_i$, we tacitly assume that the summation makes sense in $X$. Given an operator $T$, we write $A_n(T)$ or $A_n$ for the operator $(1/n)(T^n + T^{n-1} + \cdots + T + I)$. The following theorem relates the local behavior of $\sum_i a_{ni} T^i$ to the local behavior of $A_n(T)$. An $A$-matrix is one satisfying the condition (A).

**Theorem 1.** Let $T$ be a linear operator in $X$, $(a_{ni})$ an $A$-matrix, and $x$ in $X$ such that $\sum_i a_{ni} T^i x \xrightarrow{w} x$. Then there exists a constant $a$ such that $(x - ax)$ belongs to the closure of $(I - T)X$. $a$ may be chosen equal to 1 if $\lim_n \sum_i a_{ni} = 1$. $A_n(x - ax) \to 0$ if

$$\sup_n \|A_n(T)\| < \infty \quad \text{and} \quad T^n/n \to 0 \text{ strongly.}$$

**Proof.** We first prove the theorem under the additional assumption that $\sum_i a_{ni} = 1$ and $(a_{ni}) \in S$ for each $n$. Let a map $\phi: S \to X$ be defined by $\phi(a) = \sum_i a_i T^i x$. If $\sum_{i=0}^{n-1} b_i = 0$, then

$$\sum_{i=0}^{n-1} b_i T^i = b_0 (I - T) + (b_0 + b_1)(T - T^2) + (b_0 + b_1 + b_2)(T^2 - T^3) + \cdots + (b_0 + b_1 + \cdots + b_{n-1})(T^{n-1} - T^n) = P(T)(I - T),$$

where $P(T)$ is a polynomial in $T$. Therefore for each $b \in S$,

$$\phi(b) \in (I - T)\phi(S)$$

if $\sum b_i = 0$. This remark is applied to the sequence $(b_i)$ defined by $b_0 = a_n 0 - a_n$, $b_i = a_{ni}$ for $i > 0, n$ fixed. It follows that for each $n$ there exists a $y_n \in \phi(S)$ such that $\sum_i a_{ni} T^i x - x = y_n - T y_n$. Therefore $x - x$ belongs to the weak closure of $(I - T)\phi(S)$, identical (Hahn-Banach) with the strong closure $(I - T)\phi(S)$. Assume (1); $A_n(x - \overline{x}) \to 0$ follows, by approximation,
from convergence to zero of expressions of the form $T^n y/n$, $y \in X$.

Now consider the general case. Let $K$ be the set of all strictly increasing sequences of nonnegative integers. We may assume that there is a positive number $\beta$ such that $\sum_i a_{n i} > \beta$ for each $n$. (If necessary, replace $(a_{n i})$ by $(a_{k n i})$, $(k_n) \in K$; if necessary, change signs.) Let $x_i = T^i x$, $(k_n) \in K$ be such that $|\sum_i a_{n i} x_i| < 1/n$ and $\|\sum_i a_{n i} x_i\| < 1/n$. Let $d_{n i} = a_{n i}$ if $i \leq k_n$; $d_{n i} = 0$ for $i > k_n$. Then $(d_{n i})$ is an $A$-matrix, $(d_{n i}) \in S$ for each $n$ and $\lim \inf \sum_i d_{n i} = d \leq \infty$. The last $\lim \inf$ may be assumed to be limit, because $(d_{n i})$ may again be replaced by a submatrix. Set $b_{n i} = d_{n i} / \sum_j d_{n j}$ for all $n$ and $i$. Then $(b_{n i})$ is an $A$-matrix, and for each $n$, $(b_{n i}) \in S$ and $\sum b_{n i} = 1$. Furthermore, $\sum_i d_{n i} x_i \overset{w}{=} x$ implies that $\sum_i b_{n i} x_i \overset{w}{=} \alpha x$, with $\alpha = 1/d$. The first part of the proof now applies to the matrix $(b_{n i})$, showing that $(x - \alpha x) \in (I - T)X$. $\leftarrow$

Corollary 1 (Ergodic theorem of Yosida-Kakutani; cf. [4, p. 661]). Assume (1) and let $x \in X$ be such that $A_{k n} x \overset{w}{=} x$ for some $(k_n) \in K$. Then $A_n x \to x$.

Proof. $(I - T)A_{k n} x$ converges to zero (cancellation properties of Cesàro averages) and also converges weakly to $(I - T)\overline{x}$. Therefore $T\overline{x} = \overline{x}$. Write $A_{k n} x = \sum_i a_{n i} T^i x$; then $(a_{n i})$ is an $A$-matrix with $\sum_i a_{n i} = 1$; thus $A_n (x - \overline{x}) = A_n x - \overline{x} \to 0$. $\leftarrow$

Given two Banach spaces $X$, $\|\|$, and $F$, $|\|$, $F$ is said to be finitely representable in $X$, in symbols $F \overset{r}{\to} X$, iff for each finite-dimensional subspace $F'$ of $F$ and each number $\epsilon > 0$, there is an isomorphism $V$ of $F'$ into $X$ such that $|\|x - Vx\| - \epsilon$ for each $x$ in the unit ball $U_{F'}$ of $F'$. If $P$ is a property of Banach spaces, say that $X$ is super-$P$ iff $F \overset{r}{\to} X$ implies that $F$ is $P$. $X$ is called $R$-ergodic iff for each isometry $T$ on $X$ and each $x \in X$ there exists an $R$-matrix $(a_{n i})$ such that $\sum_i a_{n i} T^i x$ converges weakly. A bounded sequence $(x_n)$ in $X$ is called stable iff there is an element $\overline{x}$ such that $\|(1/n)(x_{k 1} + \cdots + x_{k n}) - \overline{x}\| \to 0$ uniformly in the set $K$. $X$ is called stable iff every bounded sequence contains a stable subsequence.
Theorem 2. A Banach space is super-stable if (and only if) it is super-R-ergodic.

Proof. In [1] and [2] the same result is proved with ergodicity (i.e., strong convergence of \( A_n(T) \) for each isometry \( T \)) instead of R-ergodicity. Assume that \( Y \) is super-R-ergodic and let \( X \not
\subset Y \). We follow the notation and argument in [1]. From a sequence \( (x_n) \) in \( U_X \) we wish to extract a stable subsequence. First obtain a subsequence \( (e_n) \) of \( (x_n) \) such that the space \( F \) generated by \( (e_n) \) and a new norm \( || \cdot || \) satisfies \( F \not
\subset X \), and also \( \frac{1}{n}(e_0 + \cdots + e_{n-1}) \rightharpoonup e \) in \( F \) implies that \( (e_n) \) contains a subsequence stable in \( X \) [1, Proposition 3]. The shift \( T \) on \( (e_n) \) is an isometry, and the norm \( || \cdot || \) is of type (IS), or "invariant under spreading" of the \( e_n \)'s [1, Lemma 1]. The (IS) property implies that, in ergodic terminology, the "tail" space is equal to the "invariant" space (strictly speaking, the space of the invariant elements). We mean by this that \( x \in \bigcap_k T^kF \) if and only if \( Tx = x \) [1, Lemma 4]. Now \( F \not
\subset X \) implies \( F \not
\subset Y \); hence \( F \) is R-ergodic: There exists an R-matrix \( (a_{ni}) \) and an element \( \bar{e} \) such that \( x_n = \sum_{i} a_{ni} e_i \rightharpoonup \bar{e} \) in \( F \). The property (B) of the matrix implies that for each \( k \), \( \bar{e} \) belongs to the weak closure of \( T^kF \), hence \( \bar{e} \in T^kF \). Indeed, if \( y_n = \sum_{i \geq k} a_{ni} e_i \), then \( y_n \in T^kF \) for each \( n \) and \( |x_n - y_n| \to 0 \), implying \( y_n \rightharpoonup \bar{e} \). Therefore \( T\bar{e} = \bar{e} \). From Theorem 1 we obtain that \( A_n e_0 \rightharpoonup \bar{e} \) in \( F \). Therefore \( (e_n) \) has a subsequence stable in \( X \). Since \( (x_n) \) is arbitrary, \( X \) is stable; since \( X \) is an arbitrary space satisfying \( X \not
\subset Y \), \( Y \) is super-stable. \( \Diamond \)

R-matrices appear in the following characterization of reflexivity.

Theorem 3. A Banach space \( X \) is reflexive if and only if for each sequence \( (x_n) \) in \( U_X \) there is an R-matrix \( (a_{ni}) \) such that \( \sum_i a_{ni} x_i \) converges weakly.

Proof. The only if part follows at once from the known characterization of reflexivity in terms of weak sequential compactness of \( U_X \). The if part is a simple consequence of a deep theorem of Pełczyński [6]; we learned about this theorem from Professor W. Johnson. Assume that \( X \) is not reflexive. By [6] there exists a bounded basic sequence \( (x_i) \) and a bounded linear functional \( f \) such that \( \limsup_i f(x_i) > 0 \). Passing to a subsequence if necessary, we may assume that there is an \( \alpha > 0 \) with \( f(x_i) \geq \alpha \) for each \( i \). Set \( y_i = x_i / f(x_i) \); then \( (y_i) \) is a bounded basic sequence and \( f(y_i) = 1 \) for each \( i \). Assume that there is an R-matrix \( (a_{ni}) \) such that \( \sum_i a_{ni} y_i \rightharpoonup \bar{y} \); let \( \bar{y} = \sum_i a_i y_i \). The continuity of the coefficient functionals and condition (B)
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imply that \( a_i = \lim_n a_{ni} = 0 \) for each \( i \). Thus \( \bar{y} = 0 \). Now

\[
0 = f(\bar{y}) = \lim_n f \left( \sum_i a_{ni} y_i \right) = \lim_n \sum_i a_{ni} f(y_i) = \lim_n \sum_i a_{ni} x_i
\]

which contradicts condition (A). 

Remarks. (1) \( A \)-matrices and \( R \)-matrices have the following properties:

Given a bounded sequence \((x_i)\), if there is an \( A \)-matrix (resp., \( R \)) \((a_{ni})\) such that \( \Sigma_i a_{ni} x_i \rightharpoonup \), then there is another \( A \)-matrix (resp., \( R \)) \((b_{ni})\) such that \( \Sigma_i b_{ni} x_i \rightarrow \) strongly. This easily follows from Mazur's theorem [4, p. 422, Corollary 14].

(2) Condition (A) (and trivially, condition (B)) cannot be dispensed with in Theorem 3: The space \( c_0 \) is alternate signs Banach-Saks: i.e., each bounded sequence contains a subsequence \((y_i)\) with

\[
(1/n)(y_1 - y_2 + \cdots + (-1)^{n-1} y_n) \to 0
\]

[3, Proposition 3.1], but \( c_0 \) is not reflexive. However, (A) and (B) may be replaced by the following slightly weaker set of conditions (R'):

(C) \( \lim_n a_{ni} = a_i \) exists for each \( i \) and \( \Sigma_i a_i \) converges, and

(D) \( \Sigma_i a_{ni} \rightharpoonup \Sigma_i a_i \), where (D) means that \( \Sigma_i a_{ni} \) exists for each \( n \) and the sequence \((\Sigma_i a_{ni})\) either diverges or converges to a limit different from \( \Sigma_i a_i \).

Theorem 3 together with the proof given above remain valid if the condition (R) is replaced by the condition (R') = (C) + (D). Thus amended, Theorem 3 includes the theorems of Nishiura and Waterman [5] and Waterman [8].

(3) A variant of the proof of Theorem 2 may be based on Remark (1) and Theorem 3, as follows: From a sequence \((x_i)\) in \( U_X \) extract a subsequence \((e_i)\) and form the space \( F \) as in Theorem 2; by \( R \)-ergodicity and Remark (1), an \( R \)-matrix \((a_{ni})\) exists such that \( \Sigma_i a_{ni} e_i \rightharpoonup \) in \( F \). Using an argument similar to that of [1, Proposition 3], one obtains a subsequence \((y_i)\) of \((e_i)\) and an \( R \)-matrix \((b_{ni})\) for which \( \Sigma_i b_{ni} y_i \rightarrow \) in \( X \). By Theorem 3, \( X \) is reflexive; since \( X \setminus Y \) was arbitrary, \( Y \) is super-reflexive.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PARIS VI, 9 QUAI ST. BERNARD, 75005, PARIS, FRANCE (Current address of A. Brunel)

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210 (Current address of L. Sucheston)

Current address (H. Fong): Department of Mathematics, Bowling Green State University, Bowling Green, Ohio 43403