PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 2, June 1975

AN ERGODIC SUPER-PROPERTY OF BANACH SPACES DEFINED BY A CLASS OF MATRICES

A. BRUNEL, H. FONG[1] AND L. SUCHESTON[1]

ABSTRACT. A matrix (a_{ni}) is called an *R-matrix* if (A) $\sum_i a_{ni} \not\to 0$, and (B) $\lim_n a_{ni} = 0$ for each i. A Banach space X is called *R-ergodic* if for each isometry T and each $x \in X$, there is an R-matrix (a_{ni}) such that $\sum_i a_{ni} T^i x \xrightarrow{w}$ (converges weakly). Given two Banach spaces F and X, write F fr X if for each finite-dimensional subspace F' of F and $\epsilon > 0$, there is an isomorphism V from F' onto a subspace of X such that $\left| \|x\| - \|Vx\| \right| < \epsilon$ for each $x \in F'$ with $\|x\| \leq 1$. X is called *super-R-ergodic* if F is R-ergodic for each F fr X.

Theorem. *X is super-R-ergodic if and only if X is super-reflexive.*

The proof is based on the following:

Theorem. *Let T be a linear operator on X, (a_{ni}) a matrix satisfying* (A), *$x \in X$ such that $\sum_i a_{ni} T^i x \xrightarrow{w} \bar{x}$. Then there is a constant α such that $(x - \alpha\bar{x}) \in \overline{(I - T)X}$.*

A matrix (a_{ni}) with real terms is called an *R-matrix* iff it satisfies the following conditions:

(A) $$\sum_i a_{ni} \not\to 0 \quad \text{as } n \to \infty;$$

(B) $$\lim_n a_{ni} = 0 \quad \text{for each } i.$$

Condition (A) means that $\sum_i a_{ni}$ exists for each n and the sequence $(\sum_i a_{ni})$ either diverges or converges to a limit different from zero.

A Banach space X is called R-ergodic iff for each isometry T and each $x \in X$ there exists an R-matrix (a_{ni}) such that $\sum_i a_{ni} T^i x$ converges weakly. It is shown that X is super-R-ergodic if and only if it is super-stable (equivalently, super-reflexive). Since R-ergodicity is clearly implied by

Presented to the Society, February 28, 1974; received by the editors June 11, 1973 and, in revised form, March 1, 1974.

AMS (MOS) subject classifications (1970). Primary 47A35; Secondary 46B10.

Key words and phrases. R-matrices, super-properties, R-ergodic, reflexive, stable.

[1] This work is supported by the National Science Foundation, grant GP 34118.

ergodicity, this is an improvement over the results of [1] and [2], to which the present theorem is reduced by an ergodic argument. It is further observed that a Banach space X is reflexive if and only if for each bounded sequence (x_i) in X there is an R-matrix (a_{ni}) such that $\Sigma_i a_{ni} x_i \xrightarrow{w}$ ($\xrightarrow{w}$ means: converges weakly).

1. A real Banach space X is given, with elements $x, y, \cdots$. Sequences of real numbers are denoted by $a = (a_i)$, $b = (b_i)$, etc. S is the set of all sequences (a_i) such that $a_i \neq 0$ for only finitely many indices i. Whenever we write $\Sigma\, a_i x_i$, we tacitly assume that the summation makes sense in X. Given an operator T, we write $A_n(T)$ or A_n for the operator $(1/n)(T^0 + T^1 + \cdots + T^{n-1})$. The following theorem relates the local behavior of $\Sigma_i a_{ni} T^i$ to the local behavior of $A_n(T)$. An *A-matrix* is one satisfying the condition (A).

Theorem 1. *Let T be a linear operator in X, (a_{ni}) an A-matrix, and x in X such that $\Sigma_i a_{ni} T^i x \xrightarrow{w} \bar{x}$. Then there exists a constant α such that $(x - \alpha\bar{x})$ belongs to the closure of $(I - T)X$. α may be chosen equal to 1 if $\lim_n \Sigma_i a_{ni} = 1$. $A_n(x - \alpha\bar{x}) \longrightarrow 0$ if*

$$\sup_n \|A_n(T)\| < \infty \quad \text{and} \quad T^n/n \longrightarrow 0 \text{ strongly.} \tag{1}$$

Proof. We first prove the theorem under the additional assumption that $\Sigma_i a_{ni} = 1$ and $(a_{ni})_i \in S$ for each n. Let a map $\phi\colon S \longrightarrow X$ be defined by $\phi(a) = \Sigma\, a_i T^i x$. If $\Sigma_{i=0}^{n-1} b_i = 0$, then

$$\begin{aligned}\sum_{i=0}^{n-1} b_i T^i &= b_0(I - T) + (b_0 + b_1)(T - T^2) + (b_0 + b_1 + b_2)(T^2 - T^3) \\ &\quad + \cdots + (b_0 + b_1 + \cdots + b_{n-1})(T^{n-1} - T^n) \\ &= P(T)(I - T),\end{aligned}$$

where $P(T)$ is a polynomial in T. Therefore for each $b \in S$,

$$\phi(b) \in (I - T)\phi(S)$$

if $\Sigma\, b_i = 0$. This remark is applied to the sequence (b_i) defined by $b_0 = a_{n0} - 1$, $b_i = a_{ni}$ for $i > 0$, n fixed. It follows that for each n there exists a $y_n \in \phi(S)$ such that $\Sigma_i a_{ni} T^i x - x = y_n - Ty_n$. Therefore $\bar{x} - x$ belongs to the weak closure of $(I - T)\phi(S)$, identical (Hahn-Banach) with the strong closure $\overline{(I - T)\phi(S)}$. Assume (1); $A_n(x - \bar{x}) \longrightarrow 0$ follows, by approximation, from convergence to zero of expressions of the form $T^n y/n$, $y \in X$.

Now consider the general case. Let $\mathbf{K}$ be the set of all strictly increasing sequences of nonnegative integers. We may assume that there is a positive number β such that $\Sigma_i a_{ni} > \beta$ for each n. (If necessary, replace (a_{ni}) by $(a_{k_n,i})$, $(k_n) \in \mathbf{K}$; if necessary, change signs.) Let $x_i = T^i x$, $(k_n) \in \mathbf{K}$ be such that $|\Sigma_{i>k_n} a_{ni}| < 1/n$ and $\|\Sigma_{i>k_n} a_{ni} x_i\| < 1/n$. Let $d_{ni} = a_{ni}$ if $i \leq k_n$; $d_{ni} = 0$ for $i > k_n$. Then (d_{ni}) is an A-matrix, $(d_{ni})_i \in S$ for each n and $\beta \leq \liminf_n \Sigma_i d_{ni} = d \leq \infty$. The last lim inf may be assumed to be *limit*, because (d_{ni}) may again be replaced by a submatrix. Set $b_{ni} = d_{ni}/\Sigma_j d_{nj}$ for all n and i. Then (b_{ni}) is an A-matrix, and for each n, $(b_{ni})_i \in S$ and $\Sigma_i b_{ni} = 1$. Furthermore, $\Sigma_i d_{ni} x_i \xrightarrow{w} \bar{x}$ implies that $\Sigma_i b_{ni} x_i \xrightarrow{w} \alpha\bar{x}$, with $\alpha = 1/d$. The first part of the proof now applies to the matrix (b_{ni}), showing that $(x - \alpha\bar{x}) \in \overline{(I-T)X}$. ⊢

Corollary 1 (Ergodic theorem of Yosida-Kakutani; cf. [4, p. 661]). *Assume* (1) *and let* $x \in X$ *be such that* $A_{k_n} x \xrightarrow{w} \bar{x}$ *for some* $(k_n) \in \mathbf{K}$. *Then* $A_n x \to \bar{x}$.

Proof. $(I-T)A_{k_n} x$ converges to zero (cancellation properties of Cesàro averages) and also converges weakly to $(I-T)\bar{x}$. Therefore $T\bar{x} = \bar{x}$. Write $A_{k_n} x = \Sigma_i a_{ni} T^i x$; then (a_{ni}) is an A-matrix with $\Sigma_i a_{ni} = 1$; thus $A_n(x - \bar{x}) = A_n x - \bar{x} \to 0$. ⊢

Given two Banach spaces X, $\|\ \|$ and F, $|\ |$, F is said to be *finitely representable* in X, in symbols F fr X, iff for each finite-dimensional subspace F' of F and each number $\epsilon > 0$, there is an isomorphism V of F' into X such that $||x| - \|Vx\|| \leq \epsilon$ for each x in the unit ball $U_{F'}$ of F'. If P is a property of Banach spaces, say that X is *super-P* iff F fr X implies that F is P. X is called *R-ergodic* iff for each isometry T on X and each $x \in X$ there exists an R-matrix (a_{ni}) such that $\Sigma_i a_{ni} T^i x$ converges weakly. A bounded sequence (x_n) in X is called *stable* iff there is an element $\bar{x}$ such that

$$\|(1/n)(x_{k_1} + \cdots + x_{k_n}) - \bar{x}\| \to 0$$

uniformly in the set $\mathbf{K}$. X is called *stable* iff every bounded sequence contains a stable subsequence.

Theorem 2. *A Banach space is super-stable if (and only if) it is super-R-ergodic.*

Proof. In [1] and [2] the same result is proved with ergodicity (i.e., strong convergence of $A_n(T)$ for each isometry T) instead of R-ergodicity. Assume that Y is super-R-ergodic and let X fr Y. We follow the notation and argument in [1]. From a sequence (x_n) in U_X we wish to extract a stable subsequence. First obtain a subsequence (e_n) of (x_n) such that the space F generated by (e_n) and a new norm $|\ |$ satisfies F fr X, and also $(1/n)(e_0 + \cdots + e_{n-1}) \to$ in F implies that (e_n) contains a subsequence stable in X [1, Proposition 3]. The shift T on (e_n) is an isometry, and the norm $|\ |$ is of type (IS), or "invariant under spreading" of the e_n's [1, Lemma 1]. The (IS) property implies that, in ergodic terminology, the "tail" space is equal to the "invariant" space (strictly speaking, the space of the invariant elements). We mean by this that $x \in \bigcap_k T^k F$ if and only if $Tx = x$ [1, Lemma 4]. Now F fr X fr Y implies F fr Y; hence F is R-ergodic: There exists an R-matrix (a_{ni}) and an element $\bar{e}$ such that $x_n = \sum_i a_{ni} e_i \xrightarrow{w} \bar{e}$ in F. The property (B) of the matrix implies that for each k, $\bar{e}$ belongs to the weak closure of $T^k F$, hence $\bar{e} \in T^k F$. Indeed, if $y_n = \sum_{i \geq k} a_{ni} e_i$, then $y_n \in T^k F$ for each n and $|x_n - y_n| \to 0$, implying $y_n \xrightarrow{w} \bar{e}$. Therefore $T\bar{e} = \bar{e}$. From Theorem 1 we obtain that $A_n e_0 \to \bar{e}$ in F. Therefore (e_n) has a subsequence stable in X. Since (x_n) is arbitrary, X is stable; since X is an arbitrary space satisfying X fr Y, Y is super-stable. ⊢

R-matrices appear in the following characterization of reflexivity.

Theorem 3. *A Banach space X is reflexive if and only if for each sequence (x_n) in U_X there is an R-matrix (a_{ni}) such that $\sum_i a_{ni} x_i$ converges weakly.*

Proof. The *only if* part follows at once from the known characterization of reflexivity in terms of weak sequential compactness of U_X. The *if* part is a simple consequence of a deep theorem of Pełczyński [6]; we learned about this theorem from Professor W. Johnson. Assume that X is *not* reflexive. By [6] there exists a bounded basic sequence (x_i) and a bounded linear functional f such that $\lim\sup_i f(x_i) > 0$. Passing to a subsequence if necessary, we may assume that there is an $\alpha > 0$ with $f(x_i) \geq \alpha$ for each i. Set $y_i = x_i / f(x_i)$; then (y_i) is a bounded basic sequence and $f(y_i) = 1$ for each i. Assume that there is an R-matrix (a_{ni}) such that $\sum_i a_{ni} y_i \xrightarrow{w} \bar{y}$; let $\bar{y} = \sum_i a_i y_i$. The continuity of the coefficient functionals and condition (B) imply that $a_i = \lim_n a_{ni} = 0$ for each i. Thus $\bar{y} = 0$. Now

$$0 = f(\bar{y}) = \lim_n f\left(\sum_i a_{ni} y_i\right) = \lim_n \sum_i a_{ni} f(y_i) = \lim_n \sum_i a_{ni},$$

which contradicts condition (A). ⊢

Remarks. (1) A-matrices and R-matrices have the following properties: Given a bounded sequence (x_i), if there is an A-matrix (resp., R-) (a_{ni}) such that $\Sigma_i a_{ni} x_i \xrightarrow{w}$, then there is another A-matrix (resp., R-) (b_{ni}) such that $\Sigma_i b_{ni} x_i \to$ strongly. This easily follows from Mazur's theorem [4, p. 422, Corollary 14].

(2) Condition (A) (and trivially, condition (B)) cannot be dispensed with in Theorem 3: The space c_0 is *alternate signs Banach-Saks*: i.e., each bounded sequence contains a subsequence (y_i) with

$$(1/n)(y_1 - y_2 + \cdots + (-1)^{n-1} y_n) \to 0$$

[3, Proposition 3.1], but c_0 is not reflexive. However, (A) and (B) may be replaced by the following slightly weaker set of conditions (R′):

(C) $\lim_n a_{ni} = a_i$ exists for each i and $\Sigma_i a_i$ converges, and

(D) $\Sigma_i a_{ni} \not\to \Sigma_i a_i$, where (D) means that $\Sigma_i a_{ni}$ exists for each n and the sequence $(\Sigma_i a_{ni})$ either diverges or converges to a limit different from $\Sigma_i a_i$.

Theorem 3 together with the proof given above remain valid if the condition (R) is replaced by the condition (R′) = (C) + (D). Thus amended, Theorem 3 includes the theorems of Nishiura and Waterman [5] and Waterman [8].

(3) A variant of the proof of Theorem 2 may be based on Remark (1) and Theorem 3, as follows: From a sequence (x_i) in U_X extract a subsequence (e_i) and form the space F as in Theorem 2; by R-ergodicity and Remark (1), an R-matrix (a_{ni}) exists such that $\Sigma_i a_{ni} e_i \to$ in F. Using an argument similar to that of [1, Proposition 3], one obtains a subsequence (y_i) of (e_i) and an R-matrix (b_{ni}) for which $\Sigma_i b_{ni} y_i \to$ in X. By Theorem 3, X is reflexive; since X fr Y was arbitrary, Y is super-reflexive.

REFERENCES

1. A. Brunel and L. Sucheston, *On B-convex Banach spaces*, Math. Systems Theory 7 (1973/74), 294–299.

2. ———, *Sur quelques conditions équivalentes à la super-réflexivité dans les espaces de Banach*, C. R. Acad. Sci. Paris Sér A-B 275 (1972), A993–A994. MR 47 #7389.

3. A. Brunel and L. Sucheston, *On J-convexity and some ergodic super-properties of Banach spaces*, Trans. Amer. Math. Soc. (to appear).

4. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.

5. T. Nishiura and D. Waterman, *Reflexivity and summability*, Studia Math. 23 (1963), 53–57. MR 27 #5107.

6. A. Pełczyński, *A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces,"* Studia Math. 21 (1961/62), 371–374. MR 26 #4156.

7. I. Singer, *Bases in Banach spaces*. I, Die Grundlehren der math. Wissenschaften, Band 154, Springer-Verlag, Berlin and New York, 1970.

8. D. Waterman, *Reflexivity and summability*. II, Studia Math. 32 (1969), 61–63. MR 39 #7399.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PARIS VI, 9 QUAI ST. BERNARD, 75005, PARIS, FRANCE (Current address of A. Brunel)

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210 (Current address of L. Sucheston)

Current address (H. Fong): Department of Mathematics, Bowling Green State University, Bowling Green, Ohio 43403