ON EULER'S CRITERION FOR CUBIC NONRESIDUES

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ABSTRACT. If \( p \) is a prime \( \equiv 1 \pmod{3} \) there are integers \( L \) and \( M \) such that \( 4p = L^2 + 27M^2 \), \( L \equiv 1 \pmod{3} \). Indeed \( L \) and \( M^2 \) are unique.

If \( D \) is a cubic nonresidue \( \pmod{p} \) it is shown how to choose the sign of \( M \) so that

\[
D^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod{p}.
\]

The case \( D = 2 \) has been treated by Emma Lehmer.

1. Introduction. If \( p \) is a prime \( \equiv 1 \pmod{3} \) there are integers \( L \) and \( M \) such that

\[
4p = L^2 + 27M^2, \quad L \equiv 1 \pmod{3}.
\]

Indeed \( L \) and \( M^2 \) are unique. Moreover, \( L, M \not\equiv 0 \pmod{p} \) so that \( 1, \) \( (L + 9M)/(L - 9M) \) and \( (L - 9M)/(L + 9M) \) are the three distinct cube roots of unity \( \pmod{p} \). Thus, if \( D \) is an integer not divisible by \( p \), by Euler's criterion we have

\[
D^{(p-1)/3} = \begin{cases} 
1, & \text{if } D \text{ is a cubic residue } \pmod{p}, \\
(L \pm 9M)/(L \mp 9M), & \text{if } D \text{ is a cubic nonresidue } \pmod{p}.
\end{cases}
\]

It is the purpose of this paper to show how the sign of \( M \) in (1.1) should be chosen so that if \( D \) is a cubic nonresidue \( \pmod{p} \) then

\[
D^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod{p}.
\]

Clearly there is no loss of generality in restricting \( D \) to be a prime \( \geq 2 \), and we consider two cases according as \( D = 2, 3 \) or \( D \geq 5 \).

The case \( D = 2, 3 \) is treated in \( \S 2 \) using the theory of cyclotomy. In this case it is well known that \( D \) is a cubic residue \( \pmod{p} \) if and only if...
In Lemma 1 explicit expressions are given for the cyclotomic numbers of order 3 (compare [2, p. 397]). These are used in conjunction with known results in the theory of cyclotomy (see Lemma 2) to show how $M$ must be specified uniquely so that (1.2) holds (Theorem 1). In Theorem 1, (a) is due to Emma Lehmer [5], and (b) is new. Her approach is different to ours.

The case $D \geq 5$ is treated in §3. In this case it is well known that if $D$ is a cubic nonresidue (mod $p$) then $LM \neq 0 \pmod D$, and use of this fact is made from time to time in the proofs. A congruence modulo $D$ (see (3.1)) for the cubic Gauss sum proved by Ankeny [1], and a criterion for cubic residuacity given by Lehmer [4], are used to show how $M$ must be specified uniquely in terms of a certain set $\mathcal{Q}_1(D)$ (see (3.6) and Lemma 5) so that (1.2) holds (Theorem 2). The set $\mathcal{Q}_1(D)$ is easy to calculate for any particular value of $D$ and the values of $\mathcal{Q}_1(D)$ are given for $D = 5, 7, 11, 13, 17, 19$.

2. $D = 2, 3$. Let $w = \exp(2\pi i/3) = \frac{1}{2}(-1 + \sqrt{-3})$, so that $1 + w + w^2 = 0$. If $p$ is a prime $\equiv 1 \pmod 3$ we set, for any $L, M$ satisfying (1.1),

$$\pi = \frac{1}{2}(L + 3M) + 3Mw,$$

so that $\pi$ is a prime factor of $p$ in the Eisenstein domain $\mathbb{Z}[w]$. We define a cubic residue character $\chi_\pi \pmod \pi$ by setting for any $a \in \mathbb{Z}[w]$, 

$$\chi_\pi(a) = \begin{cases} w^r, & \text{if } a \not\equiv 0 \pmod \pi \text{ and } a^{(p-1)/3} \equiv w^r \pmod \pi, \ 0 \leq r \leq 2, \\ 0, & \text{if } a \equiv 0 \pmod \pi. \end{cases}$$

If $g$ is a primitive root (mod $p$), so that $\chi_\pi(g) = w$ or $w^2$, for any integers $h$ and $k$ ($0 \leq h, k \leq 2$) the cyclotomic number $(h, k)_3$ of order 3 is defined to be the number of solutions $(r, s)$ of $g^{3r+h} + 1 \equiv g^{3s+k} \pmod p$ with $0 \leq r, s < (p-1)/3$. Our first lemma, which is well known, gives expressions for these cyclotomic numbers in terms of $g, L, M$ and $\pi$.

**Lemma 1.**

\[
\begin{align*}
9(0, 0)_3 &= p - 8 + L, \\
18(0, 1)_3 &= 18(1, 0)_3 = 18(2, 2)_3 = \\
&= \begin{cases} 2p - 4 - L + 9M, & \text{if } \chi_\pi(g) = w, \\ 2p - 4 - L - 9M, & \text{if } \chi_\pi(g) = w^2, \end{cases} \\
18(0, 2)_3 &= 18(2, 0)_3 = 18(1, 1)_3 = \\
&= \begin{cases} 2p - 4 - L - 9M, & \text{if } \chi_\pi(g) = w, \\ 2p - 4 - L + 9M, & \text{if } \chi_\pi(g) = w^2, \end{cases} \\
9(1, 2)_3 &= 9(2, 1)_3 = p + 1 + L.
\end{align*}
\]
For any integer \( a \neq 0 \pmod{p} \) we define the index of \( a \) with respect to \( g \), written \( \text{ind}_g(a) \), to be the unique integer \( b \) such that \( a \equiv g^b \pmod{p} \), \( 0 \leq b \leq p - 2 \).

The next lemma consists of well-known results from the theory of cyclotomy (see for example [7, Lemma 4, p. 26], and [6, Theorem 1 (\( e = 3 \)), p. 257]).

**Lemma 2.** (a) Let \( h = 0, 1, 2 \). Then \( \text{ind}_{g^h}(2) \equiv h \pmod{3} \) if and only if \( (0, h) \equiv 1 \pmod{2} \).

(b) \( \text{ind}_{g^3}(3) \equiv (0, 2)_3 - (0, 1)_3 \pmod{3} \).

As \( D \) is a cubic residue \( \pmod{p} \) if and only if \( \text{ind}_g(D) \equiv 0 \pmod{3} \), we obtain immediately from Lemmas 1, 2 and (1.1) that, for \( D = 2, 3 \), \( D \) is a cubic residue \( \pmod{p} \) if and only if \( M \equiv 0 \pmod{D} \). Thus if \( D (= 2 \text{ or } 3) \) is not a cubic residue \( \pmod{p} \) we can distinguish between the two solutions \((L, \pm M)\) of (1.1) as follows: (a) if 2 is not a cubic residue \( \pmod{p} \) then (1.1) has a unique solution \((L, M)\) satisfying \( L \equiv M \pmod{4} \), and (b) if 3 is not a cubic residue \( \pmod{p} \) then (1.1) has a unique solution \((L, M)\) satisfying \( M \equiv -1 \pmod{3} \).

We can now prove Theorem 1.

**Theorem 1.** (a) If 2 is not a cubic residue \( \pmod{p} \) and \((L, M)\) is the unique solution of (1.1) satisfying \( L \equiv M \pmod{4} \) then

\[
2^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod{p}.
\]

(b) If 3 is not a cubic residue \( \pmod{p} \) and \((L, M)\) is the unique solution of (1.1) satisfying \( M \equiv -1 \pmod{3} \) then

\[
3^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod{p}.
\]

**Proof.** (a) Let \((L, M)\) be the unique solution of (1.1) satisfying \( L \equiv M \pmod{4} \) and define \( \pi \) by (2.1). Let \( g \) be a primitive root \( \pmod{p} \), such that \( \chi_\pi(g) = \omega \). Thus for this primitive root \( g \) we have, by Lemma 1, \( 18(0, 1)_3 = 2p - 4 - L + 9M \), so that, as \( L \equiv M \pmod{4} \), we have \( (0, 1)_3 \equiv 1 \pmod{2} \).

Thus by Lemma 2(a) we have \( \text{ind}_g(2) \equiv 1 \pmod{3} \), which gives

\[
2^{(p-1)/3} \equiv \omega \pmod{\pi}.
\]

It follows from (2.1) that

\[
(L + 9M)/(L - 9M) \equiv \omega \pmod{\pi}.
\]

Putting (2.3) and (2.4) together we obtain
\[ 2^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod{n}, \]
and the required result follows as both sides are real.

(b) Let \((L, M)\) be the unique solution of (1.1) satisfying \(M \equiv -1 \pmod{3}\) and define \(n\) by (2.1). Again we choose \(g\) to be a primitive root \((\pmod{p})\) such that \(\chi_\pi(g) = \omega\), and for this primitive root we have by Lemma 1, \((0, 2)_3 = (0, 1)_3 = -M\), so that, as \(M = -1 \pmod{3}\), we have by Lemma 2(b), \(\text{ind}_g(3) \equiv \text{ind}_g(0, 2)_3 - (0, 1)_3 \equiv 1 \pmod{3}\), which gives \(3^{(p-1)/3} \equiv \omega \pmod{n}\). The rest of the proof is now the same as in (a).

Example 1. Let \(p = 37\) so that \(4p = 148 = 11^2 + 27 \cdot 1^2\). The unique solution given by Lemma 3(a) is \(L = -11, M = 1\), and the one given by Lemma 3(b) is \(L = -11, M = -1\). Thus by Theorem 1 we have
\[
2^{12} \equiv \frac{(-11) + 9(1)}{(-11) - 9(1)} = \frac{1}{10} \equiv 26 \pmod{37}
\]
and
\[
3^{12} \equiv \frac{(-11) + 9(-1)}{(-11) - 9(-1)} = 10 \pmod{37}.
\]

3. \(D \geq 5\). Let \(D\) be a prime \(\geq 5\). The Gauss sum \(G(\chi_\pi)\) is defined by
\[
G(\chi_\pi) = \sum_{n=1}^{p-1} \chi_\pi(n) \exp(2\pi in/p),
\]
and Ankeny [1] has shown that, if \(D \not\equiv p\), \(G(\chi_\pi)\) satisfies the congruence
\[
(3.1) \quad G(\chi_\pi)^{D^f} \equiv \chi_\pi(D)^{-f} \pmod{D},
\]
where \(f\) is the least positive integer such that \(D^f \equiv 1 \pmod{3}\). Using (3.1) and the well-known result \(G(\chi_\pi)^3 = p\pi\) (see for example [3, p. 116]) we obtain modulo \(D\)
\[
(3.2) \quad \chi_\pi(D) = \begin{cases} 
\frac{p^2(D-1)/3}{\pi^2(D-1)/3}, & \text{if } D \equiv 1 \pmod{3}, \\
\frac{p^{D-2}/3}{\pi(D+1)/3}, & \text{if } D \equiv 2 \pmod{3}.
\end{cases}
\]

We next define for any integer \(k\)
\[
(3.3) \quad F_D(k) = \begin{cases} 
(k^2 + 27)^{(D-1)/3}(k + 3 + 6\omega)^{(D-1)/3}, & \text{if } D \equiv 1 \pmod{3}, \\
(k^2 + 27)^{(D-2)/3}(k + 3 + 6\omega)^{(D+1)/3}, & \text{if } D \equiv 2 \pmod{3}.
\end{cases}
\]
Now Emma Lehmer [4] has shown that for any prime \(p \equiv 1 \pmod{3}\) with \(LM \neq \)}
0 (mod D), there is a set \( \mathcal{L}(D) \) depending only on D, such that D is a cubic nonresidue (mod p) if and only if \( L^2 \equiv k^2M^2 \) (mod D) for some \( k \in \mathcal{L}(D) \).

Clearly \( \mathcal{L}(D) \) may be taken as a subset of \( \{ \pm 1, \pm 2, \ldots, \pm \frac{\sqrt{D-1}}{2} \} \) and to have the property that if \( k \in \mathcal{L}(D) \) then \( -k \in \mathcal{L}(D) \). Further we may assume that for each \( k \in \mathcal{L}(D) \) there is some prime \( p \equiv 1 \) (mod 3) with \( L \equiv 0 \) (mod D) for which \( L^2 \equiv k^2M^2 \) (mod D). We also remark that \( \pm b \notin \mathcal{L}(D) \), where \( b^2 + 27 \equiv 0 \) (mod D) when \( D \equiv 1 \) (mod 3).

We prove

**Lemma 3.** If \( k \in \mathcal{L}(D) \) then

\[
F_D(k) \equiv w \pmod{D}, \quad F_D(-k) \equiv w^2 \pmod{D},
\]

or

\[
F_D(k) \equiv w^2 \pmod{D}, \quad F_D(-k) \equiv w \pmod{D}.
\]

**Proof.** As \((k + 3 + 6w)(-k + 3 + 6w) = -(k^2 + 27)\) we have

\[
F_D(k)F_D(-k) = \begin{cases} 
\frac{(k^2 + 27)^2}{2(D-1)}, & \text{if } D \equiv 1 \pmod{3}, \\
\frac{-(k^2 + 27)^{D-1}}{2}, & \text{if } D \equiv 2 \pmod{3}.
\end{cases}
\]

Since D is prime, we have \((D + 1)/3 \equiv 0 \pmod{2}\) when \( D \equiv 2 \pmod{3}\). Also as \( k^2 + 27 \not\equiv 0 \pmod{D} \) for \( k \in \mathcal{L}(D) \), we have \((k^2 + 27)^{D-1} \equiv 1 \pmod{D}\).

Hence we have

\[
F_D(k)F_D(-k) \equiv 1 \pmod{D}.
\]

Further since \( k \in \mathcal{L}(D) \) there exists a prime \( p \) for which D is a cubic nonresidue (mod p) and such that \( LM \not\equiv 0 \pmod{D} \) and \( L \equiv kM \pmod{D} \). Hence we have

\[
4p \equiv (k^2 + 27)M^2, \quad 2n \equiv (k + 3 + 6w)M \pmod{D},
\]

and so

\[
F_D(k) \equiv \begin{cases} 
\frac{(4p/M^2)^{2(D-1)/3}}{2^{2(D-1)/3}}, & \text{if } D \equiv 1 \pmod{3}, \\
\frac{(4p/M^2)^{(D-2)/3}}{2^{(D+1)/3}}, & \text{if } D \equiv 2 \pmod{3},
\end{cases}
\]

\[
\equiv \begin{cases} 
p^{2(D-1)/3} \pi^{2(D-1)/3}, & \text{if } D \equiv 1 \pmod{3}, \\
p^{(D-2)/3} \pi^{(D+1)/3}, & \text{if } D \equiv 2 \pmod{3},
\end{cases}
\]

0 (mod D), there is a set \( \mathcal{L}(D) \) depending only on D, such that D is a cubic nonresidue (mod p) if and only if \( L^2 \equiv k^2M^2 \) (mod D) for some \( k \in \mathcal{L}(D) \).

Clearly \( \mathcal{L}(D) \) may be taken as a subset of \( \{ \pm 1, \pm 2, \ldots, \pm \frac{\sqrt{D-1}}{2} \} \) and to have the property that if \( k \in \mathcal{L}(D) \) then \( -k \in \mathcal{L}(D) \). Further we may assume that for each \( k \in \mathcal{L}(D) \) there is some prime \( p \equiv 1 \) (mod 3) with \( L \equiv 0 \) (mod D) for which \( L^2 \equiv k^2M^2 \) (mod D). We also remark that \( \pm b \notin \mathcal{L}(D) \), where \( b^2 + 27 \equiv 0 \) (mod D) when \( D \equiv 1 \) (mod 3).

We prove

**Lemma 3.** If \( k \in \mathcal{L}(D) \) then

\[
F_D(k) \equiv w \pmod{D}, \quad F_D(-k) \equiv w^2 \pmod{D},
\]

or

\[
F_D(k) \equiv w^2 \pmod{D}, \quad F_D(-k) \equiv w \pmod{D}.
\]

**Proof.** As \((k + 3 + 6w)(-k + 3 + 6w) = -(k^2 + 27)\) we have

\[
F_D(k)F_D(-k) = \begin{cases} 
\frac{(k^2 + 27)^2}{2(D-1)}, & \text{if } D \equiv 1 \pmod{3}, \\
\frac{-(k^2 + 27)^{D-1}}{2}, & \text{if } D \equiv 2 \pmod{3}.
\end{cases}
\]

Since D is prime, we have \((D + 1)/3 \equiv 0 \pmod{2}\) when \( D \equiv 2 \pmod{3}\). Also as \( k^2 + 27 \not\equiv 0 \pmod{D} \) for \( k \in \mathcal{L}(D) \), we have \((k^2 + 27)^{D-1} \equiv 1 \pmod{D}\).

Hence we have

\[
F_D(k)F_D(-k) \equiv 1 \pmod{D}.
\]

Further since \( k \in \mathcal{L}(D) \) there exists a prime \( p \) for which D is a cubic nonresidue (mod p) and such that \( LM \not\equiv 0 \pmod{D} \) and \( L \equiv kM \pmod{D} \). Hence we have

\[
4p \equiv (k^2 + 27)M^2, \quad 2n \equiv (k + 3 + 6w)M \pmod{D},
\]

and so

\[
F_D(k) \equiv \begin{cases} 
\frac{(4p/M^2)^{2(D-1)/3}}{2^{2(D-1)/3}}, & \text{if } D \equiv 1 \pmod{3}, \\
\frac{(4p/M^2)^{(D-2)/3}}{2^{(D+1)/3}}, & \text{if } D \equiv 2 \pmod{3},
\end{cases}
\]

\[
\equiv \begin{cases} 
p^{2(D-1)/3} \pi^{2(D-1)/3}, & \text{if } D \equiv 1 \pmod{3}, \\
p^{(D-2)/3} \pi^{(D+1)/3}, & \text{if } D \equiv 2 \pmod{3},
\end{cases}
\]
that is

\[(3.5)\quad F_D(k) \equiv \chi_\pi(D) \pmod{D}.\]

The result now follows as \(\chi_\pi(D) = w \) or \(w^2\) since \(D\) is a cubic nonresidue \(\pmod{p}\).

Lemma 3 enables us to define for \(i = 1, 2\)

\[(3.6)\quad \mathcal{L}_i(D) = \{k \in \mathcal{L}(D) : F_D(k) \equiv w^i \pmod{D}\},\]

so that

\[\mathcal{L}_1(D) \cup \mathcal{L}_2(D) = \mathcal{L}(D), \quad \mathcal{L}_1(D) \cap \mathcal{L}_2(D) = \emptyset.

Lemma 4. Let \(D\) be a prime \(\geq 5\). If \(p\) is a prime \(\equiv 1 \pmod{3}\), for which \(D\) is a cubic nonresidue \(\pmod{p}\), then we can define \(M\) uniquely by requiring it to satisfy \(L \equiv kM \pmod{D}\) for some \(k \in \mathcal{L}_1(D)\).

Proof. As \(D\) is a cubic nonresidue \(\pmod{p}\) by Lehmer's criterion, \(L^2 \equiv k^2M^2 \pmod{D}\) for some \(k \in \mathcal{L}(D)\) and some solution \((L, M)\) of (1.1). By replacing \(k\) by \(-k\) if necessary we may assume that \(k \in \mathcal{L}_1(D)\). Now \(L \equiv \pm kM \pmod{D}\) with \(k \in \mathcal{L}_1(D)\), and as \(L\) cannot satisfy both these congruences we may choose \(M\) uniquely so that \(L \equiv kM \pmod{D}\).

We can now prove Theorem 2.

Theorem 2. Let \(D\) be a prime \(\geq 5\). If \(p\) is a prime \(\equiv 1 \pmod{3}\) for which \(D\) is a cubic nonresidue \(\pmod{p}\) and \(M\) is defined uniquely by \(L \equiv kM \pmod{D}\) for some \(k \in \mathcal{L}_1(D)\) then (1.2) holds.

Proof. It follows from (2.1) that (2.4) holds. Further as \(L \equiv kM \pmod{D}\) with \(k \in \mathcal{L}_1(D)\), we have \(F_D(k) \equiv w \pmod{D}\) and so \(\chi_\pi(D) = w \pmod{D}\), that is, \(\chi_\pi(D) = w\). Hence we have

\[D^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod{\pi},\]

and the result follows as both sides are real.

Example 2. From Lehmer's criterion for cubic residuacity [4] we deduce that

\[
\begin{align*}
\mathcal{L}(5) &= \{\pm 1, \pm 2\}, \\
\mathcal{L}(7) &= \{\pm 2, \pm 3\}, \\
\mathcal{L}(11) &= \{\pm 1, \pm 2, \pm 3, \pm 5\}, \\
\mathcal{L}(13) &= \{\pm 2, \pm 3, \pm 4, \pm 6\}, \\
\mathcal{L}(17) &= \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 6, \pm 7\}, \\
\mathcal{L}(19) &= \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 6, \pm 8\}.
\end{align*}
\]

Using (3.3) and (3.6) we obtain
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\[ \mathcal{Q}_1(5) = \{1 + 1, -2\}, \quad \mathcal{Q}_1(7) = \{1 + 2, -3\}, \]
\[ \mathcal{Q}_1(11) = \{1 - 1, -2, -3, + 5\}, \quad \mathcal{Q}_1(13) = \{-2, + 3, + 4, -6\}, \]
\[ \mathcal{Q}_1(17) = \{-1, + 2, + 4, -5, + 6, -7\}, \quad \mathcal{Q}_1(19) = \{1 + 1, + 2, -4, -5, -6, -8\}. \]

Thus Theorem 2 gives as a particular case: let \( D \) denote one of 5, 7, 11, 13, 17, 19. If \( p \) is a prime \( \equiv 1 \pmod{3} \) for which \( D \) is a cubic nonresidue \( \pmod{p} \) and \( M \) is defined uniquely by \( L \equiv kM \pmod{D} \) where

\[
k = \begin{cases} 
1 \text{ or } -2, & \text{if } D = 5, \\
2 \text{ or } -3, & \text{if } D = 7, \\
-1, -2, -3 \text{ or } 5, & \text{if } D = 11, \\
-2, 3, 4 \text{ or } -6, & \text{if } D = 13, \\
-1, 2, 4, -5, 6 \text{ or } -7, & \text{if } D = 17, \\
1, 2, -4, -5, -6 \text{ or } -8, & \text{if } D = 19,
\end{cases}
\]

then (1.2) holds.

Thus if \( p = 61 \) and \( D = 19 \) the required unique solution is \( L = 1, M = 3 \), so that

\[
19^{20} \equiv \frac{1 + 9 \cdot 3}{1 - 9 \cdot 3} = \frac{-14}{13} = 13 \pmod{61}.
\]

It is interesting to note that the sum of the elements in each of the sets \( \mathcal{Q}_1(D) \) \( (D = 5, 7, \ldots, 19) \) is congruent to \(-1 \pmod{D}\)!

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