A REAL PLACE ON THE REAL NUMBER FIELD IS TRIVIAL

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ABSTRACT. This paper proves that any real place from the real number system must be trivial. The result extends to a real place on any archimedean real closed field.

A place \( \phi \) from field \( F \) to field \( K \) is a ring homomorphism from a valuation subring \( V \) of \( F \) into \( K \) whose kernel is the unique maximal ideal of \( V \). A real field is an ordered field in which \(-1\) is not a sum of squares. A real place is a place whose codomain is a real field. A real field is quadratically closed if its positive elements are squares; it is archimedean in case for every element \( x \) there exist rational numbers \( r \) and \( s \) such that \( r < x < s \).

Let \( \mathbb{Q} \) and \( \mathbb{R} \) denote the fields of rational numbers and real numbers, respectively.

Theorem. Let \( F \) be an archimedean quadratically closed real field, and let \( K \) be any real field. If \( \phi \) is a place from \( F \) to \( K \), then \( \phi \) is an isomorphism.

Proof. We may consider \( F \) a subfield of \( \mathbb{R} \) \([1]\). Let \( V \) be the domain of \( \phi \). Since \( \text{char}(K) = 0 \), \( \mathbb{Q} \cap V \) is a valuation subring of \( \mathbb{Q} \) whose residue field has characteristic zero. Hence \( \mathbb{Q} \cap V = \mathbb{Q} \), and \( \phi \) is trivial on \( \mathbb{Q} \).

Suppose there exists \( a \in F, a \neq 0 \), such that \( \phi(a) = 0 \). We may suppose \( a > 0 \). Then there exists \( b \in \mathbb{Q}, b \neq 0 \), such that \( b^2 < a \). Then \( a - b^2 > 0 \), so there exists \( c \in F \) such that \( a - b^2 = c^2 \). Since \( a, b \in V \), we have \( c^2 \in V \); if \( c \notin V \), then \( c^{-1} \in V \) and \( c = c^{-1} \cdot c^2 \in V \); hence \( c \in V \). Then \( -\phi(b^2) = \phi(a - b^2) = \phi(c)^2 \), or \(-1 = \phi(c/b)^2 \), contradicting the fact that \( K \) is real. Hence \( \ker \phi = (0) \), and \( \phi \) is trivial on \( F \).

Corollary. A real place on \( \mathbb{R} \) is trivial.

REFERENCE

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