INEQUALITIES CONCERNING THE CHARACTERS
OF A FINITE GROUP

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ABSTRACT. Given a finite group we provide explicit bounds (in terms of the group order and numbers of conjugacy classes and involutions) for (a) the number of real valued characters of type R; (b) the sum of the degrees of the irreducible characters; (c) the sum of the entries of the character table; (d) the sums (b), (c) restricted to real valued characters. We also provide a bound on the number of elements of order $2n$ in terms of the number of elements of order $n$.

Let us first observe the following inequalities, which follow by application of the Cauchy-Schwarz inequality if one regards the summations on the left as inner products:

1. $$\sum a_{\chi} \leq c \left( \sum n_i \right)^{1/2},$$
2. $$\sum_{\chi \text{ real}} a_{\chi} \leq k_1^{1/2} \left( \sum n_i \right)^{1/2},$$
3. $$\left| \sum \chi(a) \right| \leq c^{1/2} |C(a)|^{1/2},$$
4. $$\left| \sum_{\chi \text{ real}} \chi(a) \right| \leq k_1^{1/2} |C(a)|^{1/2},$$
5. $$\sqrt{a} \leq k_1^{1/2} |C(a)|^{1/2}.$$

Here $\sqrt{a} = \sum_\chi \epsilon(\chi) \chi(a)$ is the number of solutions in $G$ of $y^2 = a$; $\epsilon(\chi) = 0, \pm 1$, depending on whether the irreducible character $\chi$ is of type $C, R, H$ (Frobenius-Schur [4]); $a_{\chi}$ is the sum of the elements in the $\chi$th row of the character table; $|C(a)| = \sum a_{\chi} \chi(a)$ is the order of the centralizer of $a \in G$ [5]; $c$ is the number of conjugacy classes; $k_1$ is the number of real conjugacy...
classes; \( n_i = |C(a_i)| \) where \( a_i \) is an element of the \( i \)th class \( (a_0 = e, n_0 = g = |G|) \); \( m_1 = \sqrt{e} \); \( l_1, l_2 = k_1 - l_1 \) are the number of characters of type \( R, H \) respectively; \( d = \sum \chi(1); d_1 = \sum_{\chi \text{ real}} \chi(1) \).

Observe that (4) and (5) both generalize a result of Brauer-Fowler, who prove a slightly stronger result when \( a = e \) \([2, \text{Theorem } 2]\), although cf. Remark 1 below; and (1) and (2) frequently provide better upper bounds on the character sums than those of \([3], [5]\).

All of these inequalities are contained in the following statement:

The real symmetric matrix

\[
M_a = \begin{bmatrix}
\sum n_i & |C(a)| & \sum \epsilon(\chi) a_x & \sum a_x & \sum a_x \\
|C(a)| & |C(a)| & \sqrt{a} & \sum \chi(a) & \sum \chi(a) \\
\sum \epsilon(\chi) a_x & \sqrt{a} & k_1 & l_1 - l_2 & l_1 - l_2 \\
\sum a_x & \sum \chi(a) & l_1 - l_2 & k_1 & k_1 \\
\sum a_x & \sum \chi(a) & l_1 - l_2 & k_1 & c
\end{bmatrix}
\]

is positive semidefinite.

For (1)—(5) merely state that certain of the \( 2 \times 2 \) principal minors obtained by permuting corresponding columns and rows of \( M_a \) are nonnegative (there are ten such minors altogether, but the other five lead to trivial or weaker inequalities than those above).

To see that \( M_a \) is positive semidefinite, merely observe that \( M_a = A_a A_a^* \) where

\[
A_a = \begin{bmatrix}
\vdots & a_x & \cdots \\
\vdots & \chi(a) & \cdots \\
\vdots & \epsilon(\chi) & \cdots \\
\vdots & \epsilon(\chi)^2 & \cdots \\
\vdots & 1 & \cdots 
\end{bmatrix}
\]

is the \( 5 \times c \) matrix whose \( \chi \)th column is as shown.

To obtain the inequalities which we desire, we permute corresponding rows and columns of \( M_a \) and compute the \( 3 \times 3 \) principal minors.

We should perhaps mention that we have here a veritable plethora of inequalities: first, for any \( a \in G \) (and permutation of corresponding columns and rows) all of the principal minors of \( M_a \) are nonnegative; we may also average the \( M_a \)'s over subsets of \( G \) to obtain positive semidefinite matrices; or we
may average the principal minors of $M_a$ over subsets of $G$; or we may use Minkowski's result that the determinant function is concave on the set of positive semidefinite matrices to obtain inequalities between the principal minors of the averaged $M_a$'s and the average of the principal minors of $M_a$. We will content ourselves here, however, with five (of the ten) $3 \times 3$ principal minors of $M_a$ when $a = e$:

(6) \[ c k_1 g + 2dm_1(l_1 - l_2) - m_1^2 c - (l_1 - l_2)^2 g - d_1^2 k_1 \geq 0, \]

(7) \[ k_1^2 g + 2d_1 m_1(l_1 - l_2) - m_1^2 k_1 - (l_1 - l_2)^2 g - d_1^2 k_1 \geq 0, \]

(8) \[ \left( \sum n_i \right) k_1 g + 2m_1 g \sum \chi a_x - m_1^2 \sum n_i - k_1 g^2 - g \left( \sum \chi a_x \right)^2 \geq 0, \]

(9) \[ \left( \sum n_i \right) k_1 g + 2d_1 g \left( \sum_{x \text{ real}} a_x \right) - d_1^2 \sum n_i - k_1 g^2 - g \left( \sum_{x \text{ real}} a_x \right)^2 \geq 0, \]

(10) \[ \left( \sum n_i \right) c g + 2d g \left( \sum a_x \right) - d^2 \sum n_i - c g^2 - g \left( \sum a_x \right)^2 \geq 0. \]

From (6), (7) we obtain (after substituting $l_2 = k_1 - l_1$)

**Theorem 1.**

\[
\begin{align*}
\frac{k_1}{2} + \frac{m_1 d_1}{2g} & & \leq \frac{1}{2g} \sqrt{(g k_1 - d_1^2)(g k_1 - m_1^2)} \\
\frac{k_1}{2} + \frac{m_1 d}{2g} & & \leq \frac{1}{2g} \sqrt{(g c - d^2)(g k_1 - m_1^2)}
\end{align*}

\[ A \]

(\[ A' \])

\[
\begin{align*}
d_1 \leq m_1 \frac{l_1 - l_2}{k_1} + \sqrt{(g k_1 - m_1^2)(k_1^2 - (l_1 - l_2)^2)} \\
d \leq m_1 \frac{l_1 - l_2}{k_1} + \sqrt{(g k_1 - m_1^2)(ck_1 - (l_1 - l_2)^2)}
\end{align*}

From (8), (9), (10), we obtain
Theorem 2.

\[ (D) \quad \sum \epsilon(\chi)a_{\chi} \leq m_1 + \left( \sum_{i \neq 0} n_i \right)^{1/2} \sqrt{k_1 - m_1^2/g}, \]

\[ (E) \quad \left| \sum_{\chi \text{ real}} a_{\chi} - d_1 \right| \leq \left( \sum_{i \neq 0} n_i \right)^{1/2} \sqrt{k_1 - d_1^2/g}, \]

\[ (F) \quad \left| \sum a_{\chi} - d \right| \leq \left( \sum_{i \neq 0} n_i \right)^{1/2} \sqrt{c - d^2/g}. \]

We conclude with three remarks and an application:

1. Concerning (A) and (A'), one may estimate \( d, d' \) by \( m_1 \leq d \leq k_1^2/2, \ g \leq d' \leq c^2/4 \) to obtain bounds on \( l_1, l_2 \) strictly in terms of \( g, m_1, k_1, c \). In particular we have \( m_1^2/g \leq l_1 \leq k_1 \) which sharpens Brauer-Fowler [2, Theorem 2] in another direction. We conjecture that \( l_1 \geq k_1/2 \); determining the exact relationship between \( l_1, l_2 \) and the internal structure \( G \) is an old unsolved problem [1]. The upper bound in (A) is attained whenever \( k_1 = l_1 \) (observe \( m_1 = d_1 \) in this case); the upper bound in (A') is attained whenever \( c = k_1 = l_1 \). Note, however, that the upper bounds are also attained for the quaternion group of order 8 (where \( l_1 = k_1 - 1 \)).

2. Concerning (B) and (C), recall \( m_1 \leq d \leq d' \) and, as in (A) and (A'), equality holds whenever \( k_1 = l_1 \) (for (B)) or \( c = k_1 = l_1 \) (for (C)). One may use the trivial estimate \( |l_1 - l_2| \leq k_1 \) to obtain bounds strictly in terms of \( g, m_1, k_1, c \).

3. Concerning (D), (E) and (F), recall that in [3] we proved that \( \Sigma \epsilon(\chi)a_{\chi} \geq (m_1 - 1) + (c - r) + (k_1 - k_2) \) so \( \Sigma a_{\chi} \geq \Sigma_{\chi \text{ real}} a_{\chi} \geq \Sigma \epsilon(\chi)a_{\chi} \geq m_1 \). We conjecture that in fact \( \Sigma a_{\chi} \geq d \) and \( \Sigma_{\chi \text{ real}} a_{\chi} \geq d_1 \). Observe that \( \Sigma a_{\chi} - d \) is the sum of the elements of the character table outside of the first column; and \( \Sigma_{\chi \text{ real}} a_{\chi} - d_1 \) is the sum of such elements in rows corresponding to real valued characters.

An application. Let us again consider the positive semidefinite matrix

\[
\begin{bmatrix}
  k_1 & \sqrt{a} \\
  \sqrt{a} & |C(a)|
\end{bmatrix}
\]

If we sum these matrices over the set of involutions of \( G \) we obtain the positive semidefinite matrix

\[
\begin{bmatrix}
  mk_1 & m_4 \\
  m_4 & gr
\end{bmatrix}
\]
where \( m \) denotes the number of involutions, \( m_4 \) the number of elements of order 4, and \( r \) the number of conjugacy classes of involutions. Hence \( m_4 \leq \sqrt{gk_1mr} \). More generally,

**Theorem 3.** If \( m_n \) denotes the number of elements of order \( n \) and \( r_n \) the number of conjugacy classes of elements of order \( n \), then

\[
\begin{align*}
m_{2n} & \leq \sqrt{gk_1m_nr_n} & \text{if } n \text{ is even,} \\
m_{2n} & \leq \sqrt{gk_1m_nr_n} - m_n & \text{if } n \text{ is odd.}
\end{align*}
\]

When \( n = 1 \) we have again the aforementioned result of Brauer-Fowler that \( m \leq \sqrt{gk_1} - 1 \). Observe also that for the quaternion group of order 8, \( m_4 = 6 \) and \( gk_1m_2r_2 = 40 \).

**BIBLIOGRAPHY**