PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 2, June 1975

A NOTE ON THE ZEROS OF FABER POLYNOMIALS

A. W. GOODMAN

ABSTRACT. By an elementary counterexample we show that a conjecture about the zeros of the Faber polynomials is false.

1. **Introduction.** Let K be a compact set consisting of more than one point and not containing the point $z = \infty$. Further, assume that the complement $\tilde{K}$ is simply-connected (on the Riemann sphere). Let $w = f(z)$ be the univalent function that maps $\tilde{K}$ onto the exterior of $|w| = 1$, normalized so that

$$f(z) = z + a_0 + a_{-1}z^{-1} + a_{-2}z^{-2} + \cdots. \tag{1}$$

This normalization can always be achieved by changing the units appropriately in the z plane, and we assume that this is done.

The polynomial part of $(f(z))^n$, denoted by $F_n(z)$, is called the Faber polynomial of degree n associated with K. According to J. L. Ullman [6], it has been conjectured that the zeros of $F_n(z)$ always lie in $H(K)$, the convex hull of K. Ullman [6] proved that the zeros of the derivative of $F_n(z)$ lie in $H(K)$, and Kövari and Pommerenke [4] have proved that if K is convex then the zeros of $F_n(z)$ lie in K. In this paper we give an elementary example that shows that if K is not convex then the zeros of $F_3(z)$ may lie outside of $H(K)$.

2. **Computation of the Faber polynomials.** Let

$$z = g(w) = w + b_0 + b_1/w + b_2/w^2 + \cdots \tag{2}$$

be the inverse function of $w = f(z)$. (Note that we have dropped the negative signs on the subscripts of b because they are unimportant.) The function $g(w)$ is univalent in $|w| > 1$ and maps that domain on $\tilde{K}$. Faber [2] proved that

$$\frac{g'(w)}{g(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}} \tag{3}$$

Received by the editors February 26, 1974.
AMS (MOS) subject classifications (1970). Primary 30A08.
Key words and phrases. Faber polynomials, zeros of Faber polynomials.

(see also Curtiss [1] or Markushevich [5, p. 106]). As Ullman [6] has observed, if we differentiate (3) with respect to z, and then integrate with respect to w we will obtain

$$\frac{1}{g(w)-z} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{nw^n}. \tag{4}$$

These two formulas, (3) and (4), and a little labor will give the first three Faber polynomials:

$$F_1(z) = z - b_0, \tag{5}$$

$$F_2(z) = z^2 - 2b_0 z - 2b_1 + b_0^2, \tag{6}$$

$$F_3(z) = z^3 - 3b_0 z^2 + 3(b_0^2 - b_1)z + 3b_0 b_1 - 3b_2 - b_0^3. \tag{7}$$

3. **The counterexample.** It is well known that if $a > 1$, then the function

$$z = \gamma(w) = w\frac{w-a}{1-aw} \tag{8}$$

maps the domain $|w| > 1$ onto the complement of a circular arc of radius 1, with end points $-e^{\pm 2i\alpha}$, where $\cos\alpha = 1/a$, and the arc contains the point $z = 1$. This is easily checked by observing that if $|w| = 1$, then $|z| = 1$, and that $\gamma'(w)$ has a simple zero at $w = e^{\pm i\alpha}$. To obtain the proper normalization we set

$$g(w) = -a\gamma(w) = w\frac{1-a/w}{1-1/aw}, \qquad a > 1, \tag{9}$$

$$g(w) = w + \frac{1-a^2}{a} + \sum_{n=1}^{\infty} \frac{1-a^2}{a^{n+1}}\frac{1}{w^n}. \tag{10}$$

Consequently $g(w)$ has the form required in equation (2) and for this function K is the arc of a circle with end points at $ae^{\pm 2i\alpha}$, and K contains the point $z = -a$. To complete the counterexample we will show that for this set K (and $a > \sqrt{3}$) the polynomial $F_3(z)$ has two zeros, z_2 and z_3, for which $|z_j|^2 > a^2$, where a is the radius of the arc K. Then z_2 and z_3 will lie outside $H(K)$.

Using the series (10) and equation (7) we find that for this K the third Faber polynomial is

$$F_3(z) = z^3 + \frac{3(a^2-1)}{a}z^2 + 3(a^2-1)z + \frac{a^6-1}{a^3}. \tag{11}$$

4. **The zeros of $F_3(z)$.** We replace z by $-a + u/a$ in two steps. First we have

$$Q_3(\zeta) \equiv a^3 F_3(\zeta/a) = \zeta^3 + 3(a^2 - 1)\zeta^2 + 3(a^2 - 1)a^2\zeta + (a^6 - 1).$$

Then, setting $\zeta = u - a^2$, we have

$$R_3(u) \equiv Q_3(u - a^2) = u^3 - 3u^2 + 3a^2u - 1. \tag{12}$$

If u_1, u_2, and u_3 are the zeros of $R_3(u)$, then $z_j = -a + u_j/a$, $j = 1, 2, 3$, are the zeros of $F_3(z)$.

Since $R_3'(u) = 3(u - 1)^2 + 3(a^2 - 1)$ and $a > 1$, the function $R_3(u)$ is strictly increasing. Further, since $R_3(0) = -1$ and $R_3(1) = 3(a^2 - 1) > 0$, it follows that $R_3(u)$ has exactly one real zero u_1 and $0 < u_1 < 1$. Further $R_3'' = 6(u - 1)$ so the graph of $R_3(u)$ is concave downward for $0 \leq u \leq 1$. The tangent to the curve at $(0, -1)$ has slope $R_3'(0) = 3a^2$. Consequently we have

$$u_1 > 1/R_3'(0) = 1/3a^2. \tag{13}$$

Since $z_j = -a + u_j/a$, we see that $F_3(z)$ has one real zero and two complex conjugate zeros. From equation (11) we have $|z_1 z_2 z_3| = (a^6 - 1)/a^3$. Consequently, since $a > 1$ and $0 < u_1 < 1$, we have

$$|z_2|^2 = |z_2 z_3| = \frac{a^6 - 1}{a^3|z_1|} = \frac{a^6 - 1}{a^3} \cdot \frac{a}{a^2 - u_1} = \frac{a^6 - 1}{a^4 - a^2 u_1}.$$

To prove that $|z_2|^2 > |a^2|$ it suffices to prove that

$$a^6 - 1 > a^2(a^4 - a^2 u_1) = a^6 - a^4 u_1.$$

This leads to $a^4 u_1 > 1$. Now suppose that $a^2 > 3$. Then using (13) we have

$$a^4 u_1 > a^4/3a^2 = a^2/3 > 1.$$

Thus if $a^2 > 3$, then the zeros z_2 and z_3 of F_3 lie outside $H(K)$.

5. **A correction.** In the first attempt to find a counterexample, the author used an arrow for the set K, since the mapping function was readily available from an earlier paper [3, p. 284]. The attempt was not successful but a review of that paper showed that equation (3.13) for the length of a barb is in error and should be replaced by

$$s_1 = 2(1 - \cos\theta_1)(1 - \gamma)^{1-\gamma}\gamma^{\gamma}. \tag{3.13*}$$

This incorrect formula has absolutely no influence on any of the theorems of that paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620