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A CORRECTION TO THE PAPER "SEMI-OPEN SETS AND SEMI-CONTINUITY IN TOPOLOGICAL SPACES" BY NORMAN LEVINE

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ABSTRACT. A subset A of a topological space is said to be *semi-open* if there exists an open set U such that $U \subseteq A \subseteq \mathrm{Cl}(U)$ where $\mathrm{Cl}(U)$ denotes the closure of U. The class of semi-open sets of a given topological space $(X, \mathcal{T})$ is denoted $\mathrm{S.O.}(X, \mathcal{T})$. In the paper *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41, Norman Levine states in Theorem 10 that if $\mathcal{T}$ and $\mathcal{T}^*$ are two topologies for a set X such that $\mathrm{S.O.}(X, \mathcal{T}) \subseteq \mathrm{S.O.}(X, \mathcal{T}^*)$, then $\mathcal{T} \subseteq \mathcal{T}^*$. In a corollary to this theorem, Levine states that if $\mathrm{S.O.}(X, \mathcal{T}) = \mathrm{S.O.}(X, \mathcal{T}^*)$, then $\mathcal{T} = \mathcal{T}^*$. An example is given which shows the above-mentioned theorem and its corollary are false. This paper shows how different topologies on a set which determine the same class of semi-open subsets can arise from functions, and points out some of the implications of two topologies being related in this manner.

In [6] Norman Levine defines a set A in a topological space X to be *semi-open* if there exists an open set U such that $U \subseteq A \subseteq \mathrm{Cl}(U)$, where $\mathrm{Cl}(U)$ denotes the closure of U. The class of semi-open sets for a given topological space $(X, \mathcal{T})$ is denoted $\mathrm{S.O.}(X, \mathcal{T})$. Levine states in Theorem 10 of [6] that if $\mathcal{T}$ and $\mathcal{T}^*$ are two topologies for a set X such that $\mathrm{S.O.}(X, \mathcal{T}) \subseteq \mathrm{S.O.}(X, \mathcal{T}^*)$, then $\mathcal{T} \subseteq \mathcal{T}^*$. In a corollary to this theorem, Levine states that if $\mathrm{S.O.}(X, \mathcal{T}) = \mathrm{S.O.}(X, \mathcal{T}^*)$, then $\mathcal{T} = \mathcal{T}^*$. The following example which is due to S. Gene Crossley and S. K. Hildebrand [1, Example 1.1] shows the above-mentioned theorem and its corollary are false.

Example. Let $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\mathcal{T}^* = \{\emptyset, \{a\}, \{a, b\}, X\}$. An exhaustion of all possibilities shows that $\mathrm{S.O.}(X, \mathcal{T}) = \mathrm{S.O.}(X, \mathcal{T}^*)$.

Crossley and Hildebrand [3] defined two topologies $\mathcal{T}$ and $\mathcal{T}^*$ on a set X to be *semi-correspondent* if $\mathrm{S.O.}(X, \mathcal{T}) = \mathrm{S.O.}(X, \mathcal{T}^*)$. It is shown in [3] that semi-correspondence is an equivalence relation on the collection of

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topologies for a set. Example 1.5 of [3] shows that it is possible to have two semi-correspondent topologies on a set X where one space is regular, completely normal, normal, T_3, T_4, T_5, paracompact, Lindelöf, and metrizable, and the other space satisfies none of these properties.

A function f from a topological space $(X, \mathfrak{T})$ into a topological space $(Y, \mathfrak{S})$ is said to be *semi-continuous* if $f^{-1}(V) \in \mathrm{S.O.}(X, \mathfrak{T})$ for every $V \in \mathfrak{S}$. We will need the following result to prove Theorem 2.

Theorem 1 [6]. *Let f be a semi-continuous function from a topological space X into a second countable topological space Y. If P is the set of discontinuities of f, then P is a set of first category.*

Before stating the next theorem, we should point out that two semi-correspondent topologies on a set X determine precisely the same nowhere dense subsets [3, Theorem 1.12], and consequently the same subsets of first category.

Theorem 2. *Let $\mathfrak{T}$ and $\mathfrak{T}^*$ be semi-correspondent topologies on a set X. If $f:(X, \mathfrak{T}) \to Y$ is semi-continuous and Y is second countable, then the set of discontinuities of f with respect to $\mathfrak{T}^*$ is a set of first category.*

Proof. Since $\mathfrak{T}$ and $\mathfrak{T}^*$ determine the same class of semi-continuous functions, the proof follows from the above remark and Theorem 1.

The following corollary is a rather striking application of Theorem 2.

Corollary. *Let $f: R \to Y$ where R is the real numbers with the usual topology and Y is second countable. If f satisfies the condition that for any $p \in R$ and any neighborhood V of $f(p)$ there exists an open set U and a nowhere dense set N such that $p \in U - N$ and $f(U - N) \subseteq V$, then the set of discontinuities of f is a set of first category.*

Proof. Example 1.6 of [3] shows that $\{U - N : U$ is open and N is nowhere dense$\}$ is the finest topology on the real numbers that has the same class of semi-open sets as the usual topology. Now apply Theorem 2.

Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces. A function $f:(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is said to be *semi-open* if for every $U \in \mathfrak{T}$, $f(U) \in \mathrm{S.O.}(Y, \mathfrak{S})$. The following two theorems show how semi-correspondent topologies can arise from functions.

Theorem 3. *Let $f:(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ be a continuous semi-open surjection. If $\mathfrak{F}$ denotes the identification topology on Y determined by f, then $\mathfrak{F}$ and $\mathfrak{S}$ are semi-correspondent.*

Proof. Since f is continuous, we have $\mathcal{S} \subseteq \mathcal{F}$ and hence $\mathrm{S.O.}(Y, \mathcal{S}) \subseteq \mathrm{S.O.}(Y, \mathcal{F})$. Let $A \in \mathcal{F}$. Now, f being semi-open and the relation $f(f^{-1}(A)) = A$ imply $A \in \mathrm{S.O.}(Y, \mathcal{S})$. Thus $\mathcal{F} \subseteq \mathrm{S.O.}(Y, \mathcal{S})$. It follows from Theorem 3 of [6] that $\mathrm{S.O.}(Y, \mathcal{F}) \subseteq \mathrm{S.O.}(Y, \mathcal{S})$ and the proof is complete.

Theorem 4. *Let $f: (X, \mathcal{T}) \to (Y, \mathcal{S})$ be a semi-continuous open injection. If $\mathcal{W}$ denotes the weak topology on X induced by f, then $\mathcal{W}$ and $\mathcal{T}$ are semi-correspondent.*

Proof. Let $f^{-1}(V) \in \mathcal{W}$ for some $V \in \mathcal{S}$. Since $f: (X, \mathcal{T}) \to (Y, \mathcal{S})$ is semi-continuous, we have $f^{-1}(V) \in \mathrm{S.O.}(X, \mathcal{T})$. Thus $\mathcal{W} \subseteq \mathrm{S.O.}(X, \mathcal{T})$ and it follows that $\mathrm{S.O.}(X, \mathcal{W}) \subseteq \mathrm{S.O.}(X, \mathcal{T})$. Now let $U \in \mathcal{T}$. The relation $U = f^{-1}(f(U))$ and f being open imply $U \in \mathcal{W}$. Hence $\mathcal{T} \subseteq \mathcal{W}$ and $\mathrm{S.O.}(X, \mathcal{T}) \subseteq \mathrm{S.O.}(X, \mathcal{W})$.

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