PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 2, June 1975

# A LOCAL CHARACTERIZATION OF DARBOUX $\mathfrak{B}$ FUNCTIONS

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ABSTRACT. A. M. Bruckner and J. B. Bruckner gave the definition of Darboux $\mathfrak{B}$ functions and proved a theorem which is a local characterization of real-valued Darboux $\mathfrak{B}$ functions. The purpose of this paper is to generalize this theorem. To this end, the definition of a function being Darboux $\mathfrak{B}$ at a point is given which has a metric continuum as its range. Hence, the theorem that a function is Darboux $\mathfrak{B}$ if and only if it is Darboux $\mathfrak{B}$ at each point.

1. **Introduction.** A function having domain and range a subset of the real line is said to be a *Darboux function* if the image of every connected subset of its domain is a connected set. A detail account of Darboux functions can be found in the survey article by Bruckner and Ceder [2].

A. M. Bruckner and J. B. Bruckner [1] gave the definition of Darboux $\mathfrak{B}$ functions which are defined on a euclidean space and having a separable metric space as the range, and proved a theorem which is a local characterization of real-valued Darboux $\mathfrak{B}$ functions. Then they made the statement that it would be of interest to know whether such a theorem exists in case the range is not a subset of the real line. The purpose of this paper is to give an answer to this query. The local condition given by Bruckner and Bruckner [1] was in terms of limit superior and limit inferior. That local condition will be replaced by one where the range of the function is a metric continuum. How much less one can assume about the range without losing the results is an open question.

§2 contains some of the needed definitions and a definition of Darboux $\mathfrak{B}$ at a point which replaces the local condition given in [1]. §3 contains the main theorem which is preceded by a lemma.

2. **Preliminaries.** In this paper $X$ will be a euclidean space, $Y$ a metric continuum, and $\mathfrak{B}$ a topological base for $X$ such that each member of $\mathfrak{B}$ is connected. A function $f$ defined on $X$ and having range $Y$ is said to be

Received by the editors February 16, 1974.
*AMS (MOS) subject classifications* (1970). Primary 54C10; Secondary 26A15.
*Key words and phrases*. Darboux functions, Darboux $\mathfrak{B}$ functions, Darboux $\mathfrak{B}$ at a point, property L.

*Darboux* $\mathfrak{B}$ provided that if $U$ is in $\mathfrak{B}$, then $f(\mathrm{cl}(U))$ is connected in $Y$. A base $\mathfrak{B}$ for $X$ is said to satisfy *condition* (*) provided any translation of an element of $\mathfrak{B}$ is in $\mathfrak{B}$. The base $\mathfrak{B}$ satisfies *condition* (**) provided that for each $x \in X$ and $U \in \mathfrak{B}, x \in \mathrm{cl}(U)$, there exists $V \in \mathfrak{B}$ such that $x \in \mathrm{cl}(V)$ and $\mathrm{cl}(V) - (x) \subset U$.

Let $f$ be a function defined on $X$ and having range $Y$. The function $f$ is said to have property L at the point $x$ in $X$ provided that, if $U \in \mathfrak{B}$ and $x$ is a boundary point of $U$, then $f(x)$ is a limit point of $f(U)$. If $x$ is a point in $X$, then a point $y$ in $Y$ is said to be *a limit point of $f$ at $x$* provided that there exists a sequence $(x_n)_{n=1}^{\infty}$ of points in $X$ which converges to $x$ such that the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $y$.

**Definition.** Let $X$ be a euclidean space and let $\mathfrak{B}$ be a base of connected sets for $X$ which satisfies conditions (*) and (**). Let $f$ be a function defined on $X$ and having range $Y$. The function $f$ is said to be *Darboux* $\mathfrak{B}$ *at the point $p$ in $X$* provided that the following statements are true.

(1) $f$ has property L at the point $p$ and $f(p)$ is a limit point of $f$ at $p$.

(2) If $U \in \mathfrak{B}, p \in U$, and $a$ and $b$ are limit points of $f$ at $p$ relative to $U$, then for each closed subset $C$ of $Y$ which separates $a$ from $b$ there exists $x \in U$ such that $f(x)$ is in $C$.

3. **A local characterization.** A lemma will be proved before proving the main theorem.

**Lemma.** *Let $f$ be a function defined on $X$ and having range $Y$ and let $\mathfrak{B}$ be a basis for $X$ which satisfies conditions* (*) *and* (**). *Let $U \in \mathfrak{B}$ and let $f(A)$ and $f(B)$ be two mutually separated sets such that the union of $A$ and $B$ is* $\mathrm{cl}(U)$. *If $f$ has property* L *at each point in $X$ and $P$ is the boundary of $A$ relative to* $\mathrm{cl}(U)$, *then the following statements are true.*

(1) *$P$ is the boundary of $B$ relative to* $\mathrm{cl}(U)$.

(2) *$P$ is closed in $X$.*

(3) *$P \cap U$ is nonempty, dense-in-itself, and of type $G_\delta$.*

(4) *$A \cap P'$ and $B \cap P'$ are both dense in $P' = P \cap U$.*

**Proof.** It is obvious that $P$ is the boundary of $B$ relative to $\mathrm{cl}(U)$ and $P$ is closed in $X$.

To prove that $P \cap U$ is nonempty, note that $P$ is nonempty; for, if $P$ were empty, then $A$ and $B$ would form a separation of the connected set $\mathrm{cl}(U)$. Suppose $P \cap U$ is empty. Then $P \subset \mathrm{cl}(U) - U$, and $U \subset A$ or $U \subset B$. Assume $U \subset A$. Choose any $x$ in $B$. Then $x$ is in $\mathrm{cl}(U) - U$ and $x$ is a boundary point of $U$. Thus $f(x)$ is a limit point of $f(U)$. So $f(x)$ is a limit point of

$f(A)$ which is in $f(B)$. This contradicts the fact that $f(A)$ and $f(B)$ are mutually separated. Therefore $P \cap U$ is nonempty.

Now suppose $P \cap U$ is not dense-in-itself. Then there is a point $x$ in $P \cap U$, say $x$ is in $P \cap A$, and a $V$ in $\mathfrak{B}$, $V \subset U$, such that $P \cap V = (x)$. If $X$ is of dimension greater than or equal to 2, then $V - (x)$ is connected. So $V - (x) \subset A$ or $V - (x) \subset B$. Since $x$ is in $P$, $V - (x)$ is not a subset of $A$. So $V - (x) \subset B$. By condition $(**)$, there is a $W \in \mathfrak{B}$ such that $x$ is in $\mathrm{cl}(W)$ and $\mathrm{cl}(W) - (x) \subset V - (x) \subset B$. Since $x$ is a boundary point of $W$, $f(x)$ is a limit point of $f(W)$. So $f(x)$ is a limit point of $f(B)$ which is in $f(A)$. This is a contradiction. Thus $P \cap U$ is dense-in-itself if $X$ is of dimension greater than or equal to 2. If the dimension of $X$ is 1, the result is easily proved.

Since $P$ is closed and $U$ is open, $P \cap U$ is of type $G_\delta$.

To prove that $A \cap P'$ is dense in $P'$, suppose that $A \cap P'$ is not dense in $P'$. Let $V \subset U$ be an open sphere centered at a point $p \in P' \cap B$ such that $V \cap A \cap P'$ is empty. Then $A \cap V$ is open and nonempty. Let $x$ be a point of $A \cap V$ such that $d(x, X - V)$ is greater than $2d(x, p)$ where $d$ is the euclidean metric of $X$. Choose $W \in \mathfrak{B}$ such that $x \in W \subset A \cap V$ and $d(\mathrm{cl}(W), X - V)$ is greater than $d(\mathrm{cl}(W), p)$. Let $q$ be a point of $B$ nearest $x$. Then $q$ is in $V$. Let $W'$ be a translation of $W$ such that $q \in \mathrm{cl}(W') - W'$ and $W' \subset A$. By condition $(*)$, $W' \in \mathfrak{B}$. Now choose $W'' \in \mathfrak{B}$ such that $q \in \mathrm{cl}(W'')$ and $\mathrm{cl}(W'') - (q) \subset W'$. Now $q \in B$ and $\mathrm{cl}(W'') - (q) \subset A$. Since $q$ is a boundary point of $W''$, we have a contradiction by the hypothesis of the theorem. Thus $A \cap P'$ is dense in $P'$. In a similar way, it can be shown that $B \cap P'$ is dense in $P'$.

**Theorem.** *Let $X$ be a euclidean space and let $\mathfrak{B}$ be a base of connected sets for $X$ which satisfies conditions $(*)$ and $(**)$. Let $f$ be a function defined on $X$ and having a metric continuum $Y$ as its range. Then $f$ is Darboux $\mathfrak{B}$ if and only if $f$ is Darboux $\mathfrak{B}$ at each point in $X$.*

**Proof.** Suppose $f$ is not Darboux $\mathfrak{B}$. Then there exists $U \in \mathfrak{B}$ such that $f(\mathrm{cl}(U))$ is not connected. Let $f(A)$ and $f(B)$ be two mutually separated sets such that the union of $A$ and $B$ is $\mathrm{cl}(U)$.

Let $Y(A)$ and $Y(B)$ be two disjoint open subsets of $Y$ such that $f(A) \subset Y(A)$ and $f(B) \subset Y(B)$. Let $C$ be the complement of $Y(A) \cup Y(B)$. Then $C$ separates $Y(A)$ and $Y(B)$. Let $P$ be the boundary of $A$ in $\mathrm{cl}(U)$ and let $P' = P \cap U$. By the lemma both the sets $A \cap P'$ and $B \cap P'$ are dense in $P'$.

We now show that if $p$ is in $A \cap U$ and $W$ is an open set containing $C$, then there exists an open set $V \in \mathfrak{B}$ containing $p$ such that if $x \in \mathrm{cl}(V)$,

then $f(x) \in Y(A) \cup W$. Suppose the condition fails for some $p \in A \cap U$ and $W \supset C$. Let $V_1, V_2, \cdots$ be a sequence of sets each containing $p$ such that $V_k \in \mathfrak{B}$ for each $k = 1, 2, \cdots$ and $\bigcap_{k=1}^{\infty} V_k = (p)$. Then for each $k$ there exists $x_k \in V_k$ such that $f(x_k)$ is not in $Y(A) \cup W$. Since $Y(A) \cup W$ is open and $Y$ is compact, $Y - Y(A) \cup W$ is compact. Thus there exists a subsequence of $(f(x_k))_{k=1}^{\infty}$ which converges to some point $y \in Y - Y(A) \cup W$, and hence $y$ is a limit point of $f$ at $p$ which is in $Y(B)$. Since $f(p) \in Y(A)$ and $f(p)$ is a limit point of $f$ at $p$, there exists $x \in U$ such that $f(x) \in C$. This is a contradiction. Similarly, if $p \in B \cap U$ and $W$ is an open set containing $C$, then there exists $V \in \mathfrak{B}$ containing $p$ such that if $x \in \mathrm{cl}(V)$, then $f(x) \in Y(B) \cup W$.

Let $W_1, W_2, \cdots$ be a sequence of open subsets of $Y$ such that $\bigcap_{k=1}^{\infty} W_k = C$. Let $p_1 \in A \cap P'$. Choose $V_1 \in \mathfrak{B}$ such that $p_1 \in V_1$, the diameter of $V_1 < 1$, $\mathrm{cl}(V_1) \subset U$, and if $x \in \mathrm{cl}(V_1)$, then $f(x) \in Y(A) \cup W_1$. Since $B \cap P'$ is dense in $P'$, there is a $p_2$ in $B \cap P' \cap V_1$. Let $V_2 \in \mathfrak{B}$ such that $p_2 \in V_2$, $V_2 \subset V_1$, the diameter of $V_2 < \frac{1}{2}$, and if $x \in \mathrm{cl}(V_2)$, then $f(x) \in Y(B) \cup W_2$. Continuing in this manner, we obtain a sequence of points $(p_k)_{k=1}^{\infty}$ and a sequence of open sets $(V_k)_{k=1}^{\infty}$ such that for each $k$, the following conditions are satisfied.

1. $p_k \in V_k \cap P'$, $V_k \in \mathfrak{B}$, $V_{k+1} \subset V_k$, the diameter of $V_k < 1/k$.
2. If $x \in \mathrm{cl}(V_k)$ and $k$ is odd, then $f(x) \in Y(A) \cup W_k$.
3. If $x \in \mathrm{cl}(V_k)$ and $k$ is even, then $f(x) \in Y(B) \cup W_k$.

Note that the set $\bigcap_{k=1}^{\infty} \mathrm{cl}(V_k)$ consists of a single point, say $p$.

Since $\mathrm{cl}(V_1) \subset U$, $p \in U$. Since $p \in \mathrm{cl}(V_k)$ for each $k = 1, 2, \cdots$, it follows that $f(p)$ is in each $W_k$. Thus $f(p) \in \bigcap_{k=1}^{\infty} W_k$, and so $f(p) \in C$. This is a contradiction, and hence $f$ is Darboux $\mathfrak{B}$.

We now assume $f$ is Darboux $\mathfrak{B}$ and prove that $f$ is Darboux $\mathfrak{B}$ at each point of $X$. The fact that $f$ has property L of the definition of Darboux $\mathfrak{B}$ at each point is obvious. We now prove that $f(p)$ is a limit point of $f$ at $p$. Let $(W_k)_{k=1}^{\infty}$ be a descending sequence of open subsets of $Y$ each containing $f(p)$ such that $\bigcap_{k=1}^{\infty} W_k = (f(p))$. Let $(V_k)_{k=1}^{\infty}$ be a descending sequence of sets in $\mathfrak{B}$ each containing $p$ such that $\bigcap_{k=1}^{\infty} V_k = (p)$. Since $f$ is Darboux $\mathfrak{B}$ and $p \in V_k$, $f(V_k)$ is a connected subset of $Y$ which contains $f(p)$ for each $k = 1, 2, \cdots$. Now there exists $x_k \in V_k$ such that $x_k \neq p$ and $f(x_k) \in W_k$ for each $k = 1, 2, \cdots$. The sequence $(x_k)_{k=1}^{\infty}$ converges to $p$ and the sequence $(f(x_k))_{k=1}^{\infty}$ converges to $f(p)$. Therefore $f(p)$ is a limit point of $f$ at $p$.

We now prove part (2) of the definition of Darboux $\mathfrak{B}$ at a point. Let $U$

$\in \mathfrak{B}$, $p \in U$, and let $a$ and $b$ be limit points of $f$ at $p$ relative to $U$. Let $C$ be a closed subset of $Y$ which separates $a$ from $b$. Thus there exists two mutually separated open sets $V$ and $W$ such that $Y - C = V \cup W$, $a \in V$ and $b \in W$. Since $a$ and $b$ are limit points of $f$ at $p$, there exist $y, z \in U$ such that $f(y) \in V$ and $f(z) \in W$. Thus $C$ separates $f(y)$ and $f(z)$. Since $f(U)$ is connected, there exists $x \in U$ such that $f(x) \in C$, and we are finished.

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