A NOTE ON JONES' FUNCTION $K$

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ABSTRACT. For each point $x$ of a continuum $M$, F. B. Jones [5, Theorem 2] defines $K(x)$ to be the closed set consisting of all points $y$ of $M$ such that $M$ is not aposyndetic at $x$ with respect to $y$. Suppose $M$ is a plane continuum and for any positive real number $\epsilon$ there are at most a finite number of complementary domains of $M$ of diameter greater than $\epsilon$. In this paper it is proved that for each point $x$ of $M$, the set $K(x)$ is connected.

A continuum $M$ (nondegenerate metric space that is compact and connected) is said to be aposyndetic at a point $p$ of $M$ with respect to a point $q$ of $M$ if there exist an open set $W$ and a continuum $H$ in $M$ such that $p \in W \subset H \subset M - \{q\}$.

Throughout this paper $S$ is the set of points of a simple closed surface (2-sphere).

Definition. Let $M$ be a continuum in $S$ and let $x$ and $y$ be distinct points of $M$. The set $S - M$ is said to be folded around $x$ with respect to $y$ if there exist two monotone descending sequences of circular regions $U_1, U_2, U_3, \ldots$ and $V_1, V_2, V_3, \ldots$ in $S$ centered on and converging to $x$ and $y$ respectively such that $\text{Cl} U_1 \cap \text{Cl} V_1 = \emptyset$ ($\text{Cl} U_1$ is the closure of $U_1$), and there exists a sequence of mutually exclusive sets $X_1, X_2, X_3, \ldots$ in $S - M$ having the following properties. For each positive integer $i$, the set $X_i$ is the union of two intersecting arc-segments (open arcs) $I_i$ and $T_i$ such that

1. $I_i \cap T_i$ is connected,
2. $I_i$ is contained in $\text{Bd} U_i$ ($\text{Bd} U_i$ is the boundary of $U_i$) and has endpoints $a_i$ and $b_i$ in $M$,
3. the sets $\text{Cl} U_{i+1}$ and ($a_i$-component of $M - V_i$) are disjoint,
Theorem. If $M$ is a continuum in $S$ and for any positive real number $\epsilon$ there are at most a finite number of complementary domains of diameter greater than $\epsilon$, then for each point $x$ of $M$, the set $K(x)$ is connected.

Proof. Assume $K(x)$ is not connected. Let $y$ be a point of $K(x)$ that does not belong to the $x$-component of $K(x)$. There exists an open disk $R$ such that $y$ belongs to $R$, the disk $\text{Cl} R$ is contained in $S - \{x\}$, and $M$ is aposyndetic at $x$ with respect to each point of $M \cap \text{Bd} R$ [6, Theorem 49, p. 17 and Theorem 13, p. 170].

Since $M$ is not aposyndetic at $x$ with respect to $y$, $S - M$ is folded around $x$ with respect to $y$ [4, Theorem 2]. Let $U_1', U_2', U_3', \ldots$, $V_1, V_2, V_3', \ldots$, and $X_1', X_2', X_3', \ldots$ be the sequences, as described in the definition, which indicate that $S - M$ is folded around $x$ with respect to $y$. Assume without loss of generality that $\text{Cl} U_1 \cap \text{Cl} R = \emptyset$ and $\text{Cl} V_1 \subset R$.

For each positive integer $n$, let $A_n$ and $B_n$ denote the $a_n$-component and the $b_n$-component of $M - R$ respectively. According to [1, Lemma and the third paragraph in the proof of Theorem 1], we can assume without loss of generality that there exist disjoint arc-segments $C$ and $E$ in $\text{Bd} R$ such that for each $n$, $A_n$ meets both $C$ and $E$ and $B_n$ meets both $C$ and $E$. For each $n$, let $c_n$ and $e_n$ be points of $A_n \cap C$ and $A_n \cap E$ respectively. Assume without loss of generality that for each $n$, $A_{n+1}$ separates $A_n$ from $A_{n+2}$ in $S - R$ [6, Theorem 28, p. 156]. For each $n$, since the arc-segment $l_n$ is contained in $S - M$, $B_{n+1}$ also separates $A_n$ from $A_{n+2}$ in $S - R$.

The sequence $c_1, c_2, c_3, \ldots$ converges to a point $v_1$ of $M \cap \text{Cl} C$ and $e_1, e_2, e_3, \ldots$ converges to a point $v_2$ of $M \cap \text{Cl} E$. The points $v_1$ and $v_2$ are distinct; for otherwise, it would follow that $M$ is not aposyndetic at $x$ with respect to $v_1$ [1, the fourth paragraph in the proof of Theorem 1].

Since $M$ is aposyndetic at $x$ with respect to each point of $\text{Bd} R$, there exist subcontinua $H_1$ and $H_2$ of $M$ and circular regions $G_1$ and $G_2$ such that $\text{Cl} G_1 \cap \text{Cl} G_2 = \emptyset$ and such that for $n = 1$ and $n = 2$, the region $G_n$ contains $v_n$, $H_n \cap \text{Cl} G_n = \emptyset$, and the point $x$ is in the interior of $H_n$ relative to $M$. There is a circular region $W$ that contains $x$ such that $\text{Cl} W \cap \text{Cl} (G_1 \cup G_2) = \emptyset$ and $W \cap M$ is contained in $H_1 \cap H_2$.

Assume without loss of generality that $\text{Cl} C$ is in $G_1$, $\text{Cl} E$ is in $G_2$,
and Cl $U_1$ is in $W$. Let $\epsilon = \text{dist}[W, R]$. Since there are at most a finite number of complementary domains of diameter greater than $\epsilon$, there exist integers $m$ and $n$ such that $T_m$ and $T_n$ belong to the same complementary domain of $M$.

Let $T'_m$ be the component of $T_m - R$ that contains $T_m \cap I_m$ and let $T'_n$ be the component of $T_n - R$ that contains $T_n \cap I_n$. Since $A_m \cup B_m \cup C \cup E$ separates $I_m$ from $R$ in $S$, we know that $T'_m$ intersects $(G_1 \cup G_2)$.

Note also that $T'_m$ intersects both $G_1$ and $G_2$, since otherwise the union of $T'_m$ and a component of $\text{Bd}(G_1 \cup G_2)$ would separate $a_m$ from $b_m$ in $S$ [6, Theorem 32, p. 181], and this would contradict the existence of $H_1$ and $H_2$. Similarly $T'_n$ intersects both $G_1$ and $G_2$.

Since $T'_m$ and $T'_n$ belong to the same complementary domain of $M$, there is an arc $A$ in $S - M$ that intersects both $T'_m$ and $T'_n$. Let $K = T'_m \cup T'_n \cup A \cup \text{Bd} G_1$ and let $H = T'_m \cup T'_n \cup A \cup \text{Bd} G_2$. The set $K \cup H$ separates $a_m$ from $b_m$ in $S$ [6, Theorem 32, p. 181]. Since $K \cap H$ is connected, we can assume without loss of generality that $K$ separates $a_m$ from $b_m$ in $S$ [6, Theorem 20, p. 173]. Since $H_1$ contains $\{a_m, b_m\}$ and misses $K$, this contradicts the fact that $H_1$ is a continuum. It follows that $K(x)$ must be connected.

As a consequence of this theorem, we have the following result announced by C. L. Hagopian in [3].

**Corollary.** $K(x)$ is connected for each point $x$ of a plane continuum that has only finitely many complementary domains.

Continua that satisfy the hypothesis of our theorem are called $E$-continua by G. T. Whyburn. In [7, Theorem 4.4, p. 113] several conditions are given that characterize local connectivity in these spaces. It is proved in [2] that semi-aposyndetic $E$-continua are arcwise connected.

**Example.** The set $K(x)$ may fail to be connected for a point $x$ of a plane continuum that is not an $E$-continuum. To see this, let $C$ be the Cantor discontinuum and define $M$ to be the quotient space

$$C \times [0, 1]/C \times \{0, 1\}.$$

Let $y$ be the separating point of $M$. Then for each point $x$ of $M - \{y\}$, the set $K(x) = \{x, y\}$.

**REFERENCES**


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