SOME UNIVERSALLY PIERCED ARCS IN $E^3$

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ABSTRACT. A subset $X$ of $E^3$ is said to be universally pierced if each 2-sphere containing $X$ can be pierced by a tame arc at each point of $X$. We show that an arc $A$ is universally pierced provided $A$ has a shrinking point $p$ such that either $p$ lies in a tame arc in $A$ or $E^3 - A$ has 1-ALG at $p$. Applying this result we show the existence of infinitely many wild universally pierced arcs.

Introduction. A subset $X$ of $E^3$ is called a pierced set if there exists a 2-sphere $S$ in $E^3$ containing $X$ such that $S$ can be pierced by a tame arc at each point of $X$. For example, a tame arc is universally pierced [5, Theorem 5] and consequently is a pierced set. The results in this paper support the following conjecture.

Conjecture. If $X$ is a pierced arc in $E^3$, then $X$ is universally pierced.

The truth of the conjecture would mean that the property of being pierced by a tame arc at a point of the arc $X$ is independent of the 2-sphere that contains $X$. In this note we give sufficient conditions, in terms of the following definitions, for an arc to be universally pierced.

A point $p$ of a set $X$ is said to be a shrinking point of $X$ if, for every open set $U$ containing $X - \{p\}$, there is an open set $V$ containing $X - \{p\}$ such that each loop in $V - X$ shrinks to a point in $U - X$. For example, the Fox-Artin arc [4] $X$, which is wild at a single point $p$, has $p$ as a shrinking point but has no other shrinking point.

The complement of an arc $X$ in $E^3$ is said to have 1-ALG (abelian local fundamental group) at a point $p$ of $X$ if, for each open set $U$ containing $p$, there is an open set $V$ such that $p \in V \subset U$ and each loop in $V - X$ which bounds homologically ($Z$ coefficients) in $U - X$ also bounds a singular disk in $U - X$. If $p$ is an endpoint of an arc $X$, then this simply says that $E^3 - X$ is 1-LC at $p$. If the complement of an arc $A$ in $E^3$ has 1-ALG at each of its...
points, then $A$ is known to be tame \cite[Theorem 3.16]{3}. In the example mentioned above \cite{4}, the complement of the arc has $1$-ALG at every point except at the shrinking point $p$.

The main theorems. The following lemma shows that to prove that a set $X$ is universally pierced it suffices to consider only 2-spheres that contain $X$ and that are locally tame modulo $X$. It follows from this lemma and \cite{9} that Rosen's special arc \cite{9} is universally pierced; however, this result also follows from Theorem 1.

**Lemma 1.** If $X$ is an arc in $E^3$ such that every 2-sphere containing $X$ that is locally tame modulo $X$ can be pierced by a tame arc at each point of $X$, then $X$ is universally pierced.

**Proof.** Let $S$ be a 2-sphere containing $X$. (If there is no such sphere, then $X$ vacuously satisfies the definition of "universally pierced".) Using Bing's side approximation theorem relative to the open set $S - X$ \cite[Theorem 4.6.5]{2} we find two 2-spheres $S_1$ and $S_2$, each locally tame modulo $X$ and lying (except for $X$ and the unions $H_1$ and $H_2$ of pairwise disjoint collections of disks in $S_1$ and $S_2$, respectively) in opposite complementary domains of $S$. Let $p$ be in $X$. By hypothesis, $S_1$ and $S_2$ can each be pierced by a tame arc at $p$. It follows \cite[Theorem 6]{5} that tame arcs $A_1$ and $A_2$ exist in $S - H_1$ and $S - H_2$, respectively, such that $p$ is an endpoint of both $A_1$ and $A_2$ and $A_1 \cap X = A_2 \cap X = \{p\}$. Thus $S$ is accessible by a tame arc from each of its complementary domains, and it follows from \cite[Theorems 2 and 3]{7} that $S$ can be pierced by a tame arc at $p$.

**Theorem 1.** If an arc $A$ in $E^3$ has a shrinking point $p$ and $E^3 - A$ has $1$-ALG at $p$, then $A$ is universally pierced.

**Remarks on Theorem 1.** The condition "$E^3 - A$ has $1$-ALG at $p$" cannot be removed since the Fox-Artin example \cite{4} satisfies the other hypotheses and is clearly not universally pierced. However, if $A$ is required to be a pierced set with a shrinking point, then $A$ ought to be universally pierced. The arc described by Alford \cite{1} satisfies these conditions but $E^3 - A$ does not have $1$-ALG at the shrinking endpoint; hence we have no proof that Alford's arc is universally pierced.

**Proof of Theorem 1.** By Lemma 1 it suffices to consider a 2-sphere $S$ containing $A$ where $S$ is locally tame modulo $A$. We first use the "$E^3 - A$ has $1$-ALG" hypothesis to show that $S$ can be pierced by a tame arc at $p$. For convenience we assume that $p$ is not an endpoint of $A$. 
Let $U'$ be an open set containing $p$, and let $D$ be a disk in $S \cap U'$ such that $p \in \text{Int } D$ and $A \cap \text{Bd } D$ is a two-point set. Let $U$ be an open set containing $p$ such that $U \cap S \subseteq \text{Int } D$ and $U \subseteq U'$, and let $C = S \cup \text{Int } S$. We shall exhibit an open set $V$ containing $p$ such that each loop in $V \cap (C - \{p\})$ is homotopic to a constant in $U' \cap (C - \{p\})$. The same argument will work for $C^* = S \cup \text{Ext } S$, so it will then follow from McMillan's characterization of piercing points [7, Theorem 1] that $S$ can be pierced by a tame arc at $p$.

Since $E^3 - S$ is 1-ulc [10, Chapter X], there must be an open set $V'$ such that $p \in V' \subseteq U$ and each loop in $V' - S$ bounds homologically in $U - S$. Now choose an open set $V$ such that $p \in V \subseteq V'$ and each loop in $V - A$ that bounds homologically in $U - A$ also contracts in $U - A$.

Let $l$ be a loop in $V \cap (C - \{p\})$. Since $l$ can be moved with a homotopy (in $C - \{p\}$) into $C$, we may assume that $l$ lies in $\text{Int } C$. Since $l$ is a loop in $V' - S$, it bounds homologically in $U - S$ and consequently in $U - A$. Then by the choice of $V$, $l$ contracts in $U - A$. Let $E$ be a singular disk in $U - A$ bounded by $l$ and notice that $E \cap S \subseteq \text{Int } D - A$. From the construction of $D$ we see that $E \cap S$ lies in the union $D'$ of the interiors of two disks in $D - A$. By Theorem 4.2.2 of [2] we may adjust $E$ to a new singular disk $E'$ where $E' \subseteq E \cup D'$, $E' \subseteq C$, and $l$ bounds $E'$. Thus $l$ contracts in $U' \cap (C - \{p\})$, as we wished to show.

We must now show that $S$ can be pierced at every point of $A$. The fact that $p$ is a point where $S$ can be pierced by a tame arc allows us to find a tame arc $B$ in $S$ such that $B \cap A = \{p\}$ [5, Theorem 6]. Let $q$ be the endpoint of $B$ different from $p$, and let $G$ be a disk in $S - A$ such that $G \cap B = \{q\}$. We shall construct a new 2-sphere $S'$ by removing the interior of the tame disk $G$ from $S$ and replacing it with a new disk $F$. Since $G$ is well away from $A$ it is clear that $S$ and $S'$ share the same piercing properties at $A$. The disk $F$ is constructed so that it is tame except at a point $x$ in $\text{Int } F$ where it is wild (see [2, Figure 3, p. 270]), $\text{Bd } F = \text{Bd } G$, and $F \cap S \subseteq G$. Since $S' = (S - G) \cup F$, it is clear that $S'$ cannot be pierced by a tame arc at the isolated wild point $x$.

Let $T$ be the union of $A \cup B$ with an arc $R$ in $F$, where $R$ has $q$ and $x$ as its endpoints. Once we show that $T$ is cellular, it will follow from [8, Theorem 3] (or from [6, Theorem 1] if $F$ is placed in the appropriate component of $E^3 - S$) that $T$ contains at most one nonpiercing point of $S'$. The theorem then follows since $x$ is a known nonpiercing point of $S'$.

To show that $T$ is cellular we let $U$ be an open set containing $T$, and we will indicate how to obtain a 3-cell $M$ in $U$ with $T \subseteq \text{Int } M$. Let $W$ be a
tame ball centered at \( p \), lying in \( U \), whose boundary intersects \( B \) at a single point; and let \( V \) be an open set containing \( A - \{ p \} \) such that loops in \( V - A \) shrink in \( U - T \) and \( V \cap (B \cup R) = \emptyset \). There is a finite collection \( \{ D_1, D_2, \ldots, D_n \} \) of disjoint disks on \( \partial U \) whose interiors cover \( A \cap \partial U \) such that each \( D_i \) lies in \( V \). Each \( \partial D_i \) bounds a singular disk \( E_i \) in \( U - T \), so \( (\partial U - \bigcup D_i) \cup (\bigcup E_i) \) is a singular 2-sphere \( H \) which does not intersect \( A \) and whose intersection with \( B \cup R \) is the point of the set \( B \cap \partial U \).

It is easy to construct a 3-cell \( Y \) lying in \( U \) with \( B \cup R \) in its interior so that \( H \cap \partial U \) is the union of three concentric simple closed curves \( J_1, J_2, J_3 \) on \( \partial U \). Then three disks are found on \( \partial Z \) whose interiors contain \( R \cap \partial Z \). These disks are then pushed along \( R \cup B \) and "over" \( p \). We add to \( \partial Y - \partial D \) three singular disks in \( H - T \) bounded by \( J_1, J_2, J_3 \) to form a singular 2-sphere \( K' \) such that \( K' \cap U \), \( K' \cap T = \emptyset \), and \( K' \) separates \( T \) from \( E^3 - U \). Now an application of the sphere theorem (see [2, Theorem 4.5.6]) yields a nonsingular polyhedral 2-sphere \( K \) with the same properties, and \( K \) bounds a 3-cell \( M \) containing \( T \) in its interior. Since \( K \) separates \( T \) from \( E^3 - U \), we see that the 3-cell \( M \) bounded by \( K \) lies in \( U \).


Theorem 2. If \( A \) is an arc with a shrinking point \( p \) and \( p \) lies in a tame subarc of \( A \), then \( A \) is universally pierced.

Proof. If \( p \) lies in the interior of a tame arc in \( A \), then Theorem 2 follows directly from Theorem 1. In any case, the proof of Theorem 1 gives the crucial steps. To see this, let \( S \) be a 2-sphere containing \( A \) and locally tame modulo \( A \), and choose a tame arc \( B \) such that \( A \cap B = \{ p \} \). Now a Fox-Artin arc [4] \( F \) is added just as in the proof of Theorem 1, and we see that \( T = A \cup B \cup R \) is cellular the same way, even though it is not necessarily an arc. Again \( T \) can contain at most the one nonpiercing point \( x \) of the sphere \( S' \). Thus \( S \) is pierced at each point of \( A \) by a tame arc.

Applications. The results of the previous section can be used to produce infinitely many wild universally pierced arcs. We begin by considering the wild arc \( A \) described by Alford [1] which has a shrinking endpoint \( p \) (by its construction) and which lies on the boundary \( S \) of a 3-cell in \( E^3 \). It is known that \( p \) is a point where \( S \) can be pierced by a tame arc; consequently there is a tame arc \( B \) on \( S \) such that \( B \cap A = \{ p \} \) [5]. Although we do not
prove that Alford's arc $A$ is universally pierced, Theorem 2 shows that the arc $A_t = A \cup B$, formed by adding a tame arc to $A$, is universally pierced. This construction gives a hint for the construction of infinitely many such arcs.

**Theorem 3.** Alford's arc with a tame tail (described above) is universally pierced.

**Theorem 4.** There exist infinitely many inequivalent universally pierced arcs in $E^3$.

**Proof.** Let $S$ be a round sphere in $E^3$, and let $X$ be an arc in $S$ with endpoints $x$ and $y$. Let $\{D_i\}$ be a null sequence of disjoint disks on $S$ such that $\{D_i\}$ converges to $\{x\}$ and $D_i \cap X$ is a spanning arc $X_i$ of $D_i$. On each $D_i$ we construct an Alford arc $A_i$ such that $A_i$ is a spanning arc of a disk $E_i$ near $D_i$ where $\text{Bd} \ E_i = \text{Bd} \ D_i$ and $E_i \cap S \subset D_i$. The construction is done so that

$$A = \left( X - \bigcup_{i=1}^{\infty} X_i \right) \cup \left( \bigcup_{i=1}^{\infty} A_i \right)$$

is an arc from $x$ to $y$. For each $n$, let

$$B_n = \left( X - \bigcup_{i=1}^{n} X_i \right) \cup \left( \bigcup_{i=1}^{n} A_i \right).$$

Each $A_i$ has a shrinking endpoint $p_i$, and we insist that $A_i$ be constructed so as to have $p_i$ the first point of $A_i \cap B_n$ ($i \leq n$) when $x$ is considered the first point of $B_n$. Then $p_i$ lies in a tame arc $T_i$ in $B_n$, and the arc $A_i \cup T_i$ is universally pierced by Theorem 3. Since $B_n$ is the union of universally pierced arcs, it is also universally pierced. Of course $B_n$ and $B_m$ ($n \neq m$) are not equivalently embedded in $E^3$ since the wild set in $B_n$ has $n$ components while $B_m$'s wild set has $m$ components.

**Questions.** We believe that all of the following questions have affirmative answers. Three of these questions are merely special cases of the conjecture stated earlier.

1. If an arc $A$ is a pierced set and $A$ is cellular, then is $A$ universally pierced?
2. Is Alford's arc $[1]$ universally pierced? This question is equivalent to one of Rosen's by Lemma 1.
3. If an arc $A$ is a pierced set and $A$ has a shrinking point, then is $A$ universally pierced?
4. Is every pierced nondegenerate continuum universally pierced?

REFERENCES


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