TEST MODULES
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ABSTRACT. The results of this paper arose from an investigation of the class of $\Sigma$-modules, i.e., those modules $M$ for which $\text{Hom}_R(M, -)$ commutes with direct sums. A module $T$ is called a test module if $\text{Hom}_R(M, -)$ commutes with direct sums of copies of $T$ only when $M$ is a $\Sigma$-module. Test modules are characterized and their relation to cogenerators is investigated.

Throughout $N$ will denote the set of natural numbers, $R$ will denote an associative ring with identity, and module will mean unitary left $R$-module. For modules $L$ and $M$ and indexing set $I$, $L^{(I)}$ will denote the direct sum of $|I|$ copies of $L$ and, for convenience, $\text{Hom}_R(M, L)$ will be written $\text{Hom}(M, L)$.

The modules $M$ for which $\text{Hom}(M, -)$ commutes with direct sums have been called $\Sigma$-modules by Rentschler [5]. A systematic study of $\Sigma$-modules is given in his thesis [4]. $\Sigma$-modules have been considered by at least three other authors [1, p. 54], [2], and [3].

It follows from the definition that $M$ is a $\Sigma$-module if and only if, for each family of modules $\{L_i | i \in I\}$ and for each $R$-homomorphism $f: M \rightarrow \bigoplus_{i \in I} L_i$, $\pi_i f = 0$ for all but a finite number of $i \in I$. We will consistently use $\pi_i: \bigoplus_{i \in I} L_i \rightarrow L_i$ to denote the obvious projection map. It is possible to place certain restrictions on the families $\{L_i | i \in I\}$ which must be considered. It is only necessary to consider families, each of whose members is an injective module; the indexing set $I$ may be taken to be countable. The following theorem gives a further reduction which is useful.

**Theorem 1.** A module $M$ is a $\Sigma$-module if and only if, for each module $L$, $\text{Hom}(M, -)$ commutes with direct sums of the module $L$.

**Proof.** The "only if" part is trivial. For the "if" part, begin with a family $\{L_i | i \in I\}$ of modules; set $L = \bigoplus_{i \in I} L_i$; and let $\mu_i: L^{(I)} \rightarrow L$ denote the projection map. Now let $f \in \text{Hom}(M, L)$ and define $\overline{f}: M \rightarrow \bigoplus_{i \in I} L_i$. The result follows. 

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L(\text{via}(\tau_f)(m)) = y \in L$, where $y = 0$ if $i \neq j$ and $y = (\tau_f)(m)$ if $i = j$. 

$f$ is a homomorphism and the assumption yields a finite subset $J$ of $I$ such that if $j \in I - J$, $(\mu_J)(M) = 0 \in L$. If $(\tau_f)(M) \neq 0$, then $(\tau_f)(M) \neq 0$ so $(\mu_J)(M) \neq 0$ and it follows that $i \in J$. This shows that $M$ is a $\Sigma$-module.

Remark. It can be shown that one need consider only countable direct sums of the various modules $L$.

This theorem suggests the question: Is there one module $T$ so that if $\text{Hom}(M, -)$ commutes with direct sums of $T$ then $M$ is a $\Sigma$-module? Such a module $T$ would serve as a "test module" for $\Sigma$-modules. In fact we adopt this as our definition of a test module. We will show next that test modules (always) exist and are quite familiar modules.

**Theorem 2.** A module $T$ is a test module if and only if, for each module $X \neq 0$, $\text{Hom}(X, T) \neq 0$.

**Proof.** Suppose $T$ is a test module and $\text{Hom}(X, T) = 0$ for a module $X$. Then $\text{Hom}(X^{(N)}, T) = 0$ so $X^{(N)}$ is a $\Sigma$-module. This is impossible if $X \neq 0$. Thus $X = 0$.

Conversely, suppose $T$ is a module satisfying: For each module $X \neq 0$, $\text{Hom}(X, T) \neq 0$. Further assume that $X$ is a module such that $\text{Hom}(X, -)$ commutes with direct sums of $T$. We must show that $X$ is a $\Sigma$-module. Consider any module $L$ and $f \in \text{Hom}(X, L^{(N)})$. Assume, by way of contradiction, that the set $K = \{n \mid n \in N$ and $(p_n)(X) \neq 0\}$ is an infinite set, where $p_n : L^{(N)} \rightarrow L$ is the $n$th projection. For each $k \in K$, select $0 \neq h_k \in \text{Hom}(p_n f(X), M)$. If $n \in N$ and $n \notin K$ let $h_n = 0 : p_n f(X) \rightarrow M$. If $k \in K$ there exists $x_k \in X$ such that $h_k(p_k f(x_k)) \neq 0$. Now put $h = \bigoplus_{n \in N} h_n : \bigoplus_{n \in N} p_n f(X) \rightarrow M^{(N)}$. One easily checks that $hf \in \text{Hom}(X, M^{(N)})$. Showing $\pi_k(hf) \neq 0$ if $k \in K$ will contradict the fact that $\text{Hom}(X, -)$ commutes with direct sums of $T$.

Let $k \in K$,

$$hf(x_k) = h(f(x_k)) = h((p_n f(x_k))) = (\bigoplus h_n)(p_n f(x_k)) = (h_n p_n f(x_k))) .$$

From above the $k$th component is nonzero. Thus the $k$th projection of $hf$ is nonzero. With the help of Theorem 1, this completes the proof.

**Corollary.** A cogenerator (for the category of left $R$-modules) is a test module.

This shows, in answer to the question above, that test modules (always) exist but it raises another question. When is a test module a cogenerator? Before giving the answer we require the following fact.

**Lemma.** For a module $M$ there is a submodule $H$ of $M$ and a simple
module \( S \) such that \( M/H \) can be embedded in \( I(S) \), the injective hull of \( S \).

**Proof.** Choose \( K \subseteq L \subseteq M \) with \( L/K \) simple. If \( L/K \subseteq M/K \) is not essential, choose \( H/K \subseteq M/K \) such that \( H/K \cap L/K = 0 \) and \( H/K \) is maximal with respect to this property. Then \( (L+H)/H \) is simple and essential in \( M/H \).

The next theorem may be of independent interest.

**Theorem 3.** For a ring \( R \) the following are equivalent:

(a) every test module is a cogenerator;

(b) for each simple module \( S \), and each submodule \( L \subseteq I(S) \), \( I(S)/L \) contains an isomorphic copy of \( I(S) \).

**Proof.** Assume (b) holds. Let \( C \) be a test module and consider a simple module \( S \neq 0 \). By Theorem 2 we choose \( 0 \neq f \in \text{Hom}(I(S), C) \). By hypothesis \( I(S) \subseteq I(S)/\text{Ker } f \subseteq C \) so \( C \) is a cogenerator.

Now assume (b) fails. Then for some simple module \( S \), we have \( N \subseteq I(S) \) such that \( I(S)/N \) does not contain a copy of \( I(S) \). Let

\[
C = (I(S)/N) \oplus \left( \bigoplus \{ I(U) \mid U \text{ is simple and } U \not\cong S \} \right) \oplus \left( \bigoplus \{ M \mid M \subseteq I(S) \} \right).
\]

\( C \) does not contain a copy of \( I(S) \) so is not a cogenerator. However, we will show that \( C \) is a test module by using Theorem 2.

Let \( X \neq 0 \) be a module. By the Lemma we choose a simple module \( U \) such that \( X/Y \subseteq I(U) \) for some submodule \( Y \subseteq X \). If \( U \cong S \) then, trivially, \( \text{Hom}(X, C) \neq 0 \). We consider the two cases:

1. \( X/Y \cong I(S) \),
2. \( X/Y \subset I(S) \), but \( X/Y \not\cong I(S) \).

In the first case, use \( I(S)/N \) to get the nonzero element of \( \text{Hom}(X, C) \); and, in the second case, use one of the \( M \)'s, \( M \subseteq I(S) \). This completes the proof.

The authors would like to thank Professor E. Enochs for the clever construction in the proof of Theorem 3. We note that Tiwary [6] and Vamos [7] have shown that, over an integral domain \( R \), \( I(S) \cong I(S)/K \) for all simple modules \( S \) and all submodules \( K \subseteq I(S) \), if and only if, \( R_p \) is a PID for all prime ideals \( P \) of \( R \). Thus, for example, over a Dedekind domain a test module is a cogenerator.

The condition (b) of Theorem 3 appears to be interesting. Among the things it implies are: The socle of \( I(S)/K \), \( K \subseteq I(S) \), consists of copies of \( S \) and is essential in \( I(S)/K \).

**REFERENCES**


