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# A REDUCTION OF THE FUNDAMENTAL CONJECTURE ABOUT LOCALLY COMPACT ANR'S

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ABSTRACT. In this note, we show that the conjecture: *the product of a locally compact ANR by the Hilbert cube is a Hilbert cube manifold*, can be reduced to its compact AR case, i.e.: *the product of a compact AR by the Hilbert cube is a Hilbert cube manifold.*

1. The school of R. D. Anderson formulated the following fundamental conjecture [1, p. 14]:

(C) *If $X$ is a locally compact first countable* (*metrizable*) *ANR, then $X \times Q$ is a $Q$-manifold.*[1]

Recall that a $Q$-manifold is a first countable metrizable space, in which every point has a neighborhood homeomorphic to an open set of the Hilbert cube $Q = I^{\infty} = [-1, +1]^{\infty}$. An affirmative answer to (C) would imply (among other things) the finiteness of homotopy type of compact ANR's [6].

By taking the mapping cylinder of a proper embedding of $X$ into $Q \times [0, 1)$, Henderson [4] observed that (C) is implied by:

(C′) *If $X$ is a locally compact first countable* (*metrizable*) *AR, then $X \times Q$ is a $Q$-manifold.*

Our goal is to show that (C) is equivalent to the following particular case (which may be simpler!):

(C″) *If $X$ is a compact metrizable AR, then $X \times Q$ is a $Q$-manifold.*

We recall that a continuous map is called *proper*, if inverse images of compact sets are compact sets.

If $X$ is a locally compact space, by $X_+$ we mean the one point compactification of $X$.

We wish to thank T. A. Chapman for a very useful discussion, which revealed that Lemma 2.1 is the key fact to establish.

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[1]The analogue for $l^2$ in place of $Q$ has recently been proved by Toruńczyk [7], [8].

2. We prove here the key lemma.

**Lemma 2.1.** *Let $Y$ be a locally compact first countable metrizable space, and $X \subset Y$ a retract of $Y$. Then $X \times [0, \infty)$ is a proper retract of $Y \times [0, \infty)$.*

**Proof.** Let $d$ be a metric on $Y$, for which the bounded closed sets are the compact subsets of $Y$. Denoting by $r: Y \to X$ a retraction of $Y$ on $X$, we define a closed neighborhood $V$ of $X$ by $V = \{x \in Y;\ d(r(x), x) \leq 1\}$. Clearly $r|V$ is proper.

Choose a proper map $\phi$: $Y - \operatorname{Int} V \to [0, \infty)$. Then, by observing that $X$ is closed in $Y$, we can use Tietze-Urysohn extension theorem to find an extension $\overline{\phi}$ of $\phi$, such that $\overline{\phi}|X = 0$.

We then define $\overline{r}: Y \times [0, \infty) \to X \times [0, \infty)$ by $\overline{r}(y, t) = (r(y), t + \overline{\phi}(y))$. One can easily check that $\overline{r}$ is a proper retraction of $Y \times [0, \infty)$ onto $X \times [0, \infty)$. □

**Proposition 2.2.** *If $X$ is a locally compact first countable (metrizable) AR, the one point compactification of $X \times [0, \infty)$ is a compact AR.*

**Proof.** $X$ can be embedded as a closed set of $Q \times [0, 1)$, and by the AR property, we can consider $X$ as a retract of $Q \times [0, 1)$.

If we apply the preceding lemma, we see that $(X \times [0, \infty))_+$ is a retract of $T = (Q \times [0, 1) \times [0, \infty))_+$. But $T$ is homeomorphic to the cone on $Q$, hence to $Q$ by [3, p. 14]. Hence $(X \times [0, \infty))_+$ is an AR, being the retract of the compact AR $Q$. □

3. (C$''$) implies (C$'$).

**Proof.** Supposing that (C$''$) is true, let $X$ be a locally compact first countable AR. By the preceding proposition, we obtain that $X \times [0, \infty) \times Q$ is a $Q$-manifold. Observe now that $X \times Q$ is a $Q$-manifold, since $X \times [0, 1] \times Q$ is the union of two open sets which are $Q$-manifolds, namely $X \times [0, 1) \times Q$ and $X \times (0, 1] \times Q$. □

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