A REDUCTION OF THE FUNDAMENTAL CONJECTURE ABOUT
LOCALLY COMPACT ANR'S

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ABSTRACT. In this note, we show that the conjecture: the product of a
locally compact ANR by the Hilbert cube is a Hilbert cube manifold, can
be reduced to its compact AR case, i.e.: the product of a compact AR by
the Hilbert cube is a Hilbert cube manifold.

1. The school of R. D. Anderson formulated the following fundamental
conjecture [1, p. 14]:

(C) If X is a locally compact first countable (metrizable) ANR, then
X x Q is a Q-manifold.\footnote{The analogue for l^2 in place of Q has recently been proved by Toruńczyk [7], [8].}

Recall that a Q-manifold is a first countable metrizable space, in which
every point has a neighborhood homeomorphic to an open set of the Hilbert
cube Q = l^\infty = [-1, + 1]^\infty. An affirmative answer to (C) would imply (among
other things) the finiteness of homotopy type of compact ANR's [6].

By taking the mapping cylinder of a proper embedding of X into Q x
[0, 1), Henderson [4] observed that (C) is implied by:

(C') If X is a locally compact first countable (metrizable) AR, then
X x Q is a Q-manifold.

Our goal is to show that (C) is equivalent to the following particular
case (which may be simpler!):

(C'') If X is a compact metrizable AR, then X x Q is a Q-manifold.

We recall that a continuous map is called proper, if inverse images of
compact sets are compact sets.

If X is a locally compact space, by X_+ we mean the one point com-
pactification of X.

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2. We prove here the key lemma.

Lemma 2.1. Let $Y$ be a locally compact first countable metrizable space, and $X \subseteq Y$ a retract of $Y$. Then $X \times [0, \infty)$ is a proper retract of $Y \times [0, \infty)$.

Proof. Let $d$ be a metric on $Y$, for which the bounded closed sets are the compact subsets of $Y$. Denoting by $r: Y \rightarrow X$ a retraction of $Y$ on $X$, we define a closed neighborhood $V$ of $X$ by $V = \{x \in Y; d(r(x), x) \leq 1\}$. Clearly $r|V$ is proper.

Choose a proper map $\phi: Y - \text{Int } V \rightarrow [0, \infty)$. Then, by observing that $X$ is closed in $Y$, we can use Tietze-Urysohn extension theorem to find an extension $\overline{\phi}$ of $\phi$, such that $\overline{\phi}|X = 0$.

We then define $\overline{r}: Y \times [0, \infty) \rightarrow X \times [0, \infty)$ by $\overline{r}(y, t) = (r(y), t + \overline{\phi}(y))$. One can easily check that $\overline{r}$ is a proper retraction of $Y \times [0, \infty)$ onto $X \times [0, \infty)$. □

Proposition 2.2. If $X$ is a locally compact first countable (metrizable) AR, the one point compactification of $X \times [0, \infty)$ is a compact AR.

Proof. $X$ can be embedded as a closed set of $\mathbb{Q} \times [0, 1)$, and by the AR property, we can consider $X$ as a retract of $\mathbb{Q} \times [0, 1)$.

If we apply the preceding lemma, we see that $(X \times [0, \infty))_+ \subseteq T = (\mathbb{Q} \times [0, 1) \times [0, \infty))_+$. But $T$ is homeomorphic to the cone on $\mathbb{Q}$, hence to $\mathbb{Q}$ by [3, p. 14]. Hence $(X \times [0, \infty))_+$ is an AR, being the retract of the compact AR $\mathbb{Q}$. □

3. (C") implies (C').

Proof. Supposing that (C") is true, let $X$ be a locally compact first countable AR. By the preceding proposition, we obtain that $X \times [0, \infty) \times Q$ is a $Q$-manifold. Observe now that $X \times Q$ is a $Q$-manifold, since $X \times [0, 1] \times Q$ is the union of two open sets which are $Q$-manifolds, namely $X \times [0, 1] \times Q$ and $X \times (0, 1] \times Q$. □

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