SEMILATTICES ON PEANO CONTINUA

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ABSTRACT. A continuum is cell-cyclic if every cyclic element is an n-cell for some integer n. It is shown that every cell-cyclic Peano continuum admits a topological semilattice.

By a semigroup we mean a Hausdorff topological space together with an associative multiplication. One of the oldest problems in semigroup theory is: "Given a space X with topological properties P, does X admit the structure of a semigroup having algebraic properties Q?". In the case when Q is "commutative and idempotent", X is said to be a semilattice, and another approach is open. If one can define on X a partial order so that the operation \( \wedge : X \times X \rightarrow X \) defined by \( \wedge(a, b) = \text{l.u.b.}\{a, b\} \) is continuous, then \((X, \wedge)\) is a semilattice. Knight has shown [3] that any Peano (locally connected metric) continuum admits a partial order with closed graph.

In order for a Peano continuum to be a semilattice it must be acyclic [6], however not all acyclic Peano continua admit a semilattice structure. Lawson and the author have shown [5] that any semilattice on a finite-dimensional Peano continuum which is not one-dimensional contains a two-cell. Thus the example given by Borsuk in [1] is a two-dimensional Peano continuum which does not admit a semilattice. We prove here that a Peano continuum every cyclic element of which is an n-cell for some integer n admits a semilattice.

We shall use the cyclic element notation and results of Whyburn [7] and Kuratowski and Whyburn [4], slightly modified in the following way. In a Peano continuum X, we say x separates a and b if each arc from a to b contains x, and a cyclic element D separates a and b if each arc from a to b meets D. \( E(a, b) \) denotes the set of points which separate a and b (including a and b) and is a compact partially ordered set under the ordering \( x \preceq y \) iff x separates y and a. \( C(a, b) \) denotes the cyclic chain from
a to b and is \{x: \text{ some arc from } a \text{ to } b \text{ contains } x\}. Given a point a and a cyclic chain C(p, q), if a \notin C(p, q) there is a unique element x of C(p, q) such that x separates each element of C(p, q) from a. Denote x by P(C(p, q), a). If a \in C(p, q), set P(C(p, q), a) = a. We will use I to denote the unit interval under min multiplication.

**Lemma 1** ([4, p. 70]). Let X be a Peano continuum, and let C be a fixed cyclic chain of X. The function \( f_c: X \to C \) defined by \( f_c(x) = P(C, x) \) is a monotone retraction mapping \( X \to C \) into the boundary of C.

Note that since a cyclic element is a cyclic chain between any pair of its points, the above holds for cyclic elements as well.

**Lemma 2.** Let a be an element of a cyclic chain C in a Peano continuum X, and let \( \epsilon > 0 \). There exists \( \delta > 0 \) such that for each x in X - C, \( P(C, x) \) in \( B(a, \delta) - \{a\} \) (the deleted open ball about a) implies x is in \( B(a, \epsilon) \), and also x in \( B(a, \delta) \) implies \( P(C, x) \) is in \( B(a, \epsilon) \).

**Proof.** If a is in the interior of C, choose \( \delta < \epsilon \) such that \( B(a, \delta) \subseteq C^0 \). Otherwise the components of X - C form a null sequence at most [7], so there are a finite number of diameter \( > \epsilon/2 \). Choose a point \( x_i \) in each, and let

\[
\delta < \min \{|d(a, P(C, x_i))|d(a, P(C, x_i)) > 0| \cup |\epsilon/2\}.
\]

Then if \( P(C, y) \) is in \( B(a, \delta) - \{a\} \), either \( y \in C \cap B(a, \delta) \subseteq B(a, \epsilon) \) or y is in a component of X - C of diameter \( < \epsilon/2 \). Thus

\[
d(y, a) \leq d(y, P(C, y)) + d(P(C, y), a) < \epsilon/2 + \epsilon/2,
\]

so the first implication holds. The second follows from Lemma 1.

The semilattice structure on X. A Peano continuum X is cell-cyclic iff every true cyclic element of X is an n-cell for some integer n. Let \( K_n \) be \( l^n \) under coordinatewise multiplication and \( K'_n \) be \( T^n \) under coordinatewise multiplication, where T is the subsemilattice \( l \times \{0\} \cup \{0\} \times l \) of \( l^2 \). Then \( K_n \) and \( K'_n \) are semilattices on the n-cell, with the former having its minimum element on the boundary and the latter having its in the interior. Fix an element 0 of X.

For each cyclic element D of X define a function \( b_D \) as follows: If \( P(D, 0) \) is on the boundary (interior) of D let \( b_D: K_n (K'_n) \to D \) be a homeomorphism mapping the minimum element of \( K_n (K'_n) \) to \( P(D, 0) \).
Lemma 3. Let $X$ be a cell-cyclic Peano continuum with $a$, $b$, and $c$ in $X$. Suppose no cyclic element containing one of $b$ and $c$ separates the other from $a$. Then there exists a (possibly degenerate) cyclic element which separates any two of $a$, $b$, and $c$.

Proof. Case I. Some cyclic element containing one element separates the other two. Then this cyclic element separates any two of $a$, $b$, and $c$.

Case II. No cyclic element containing one of $a$, $b$, and $c$ separates the other two. This implies no two of $a$, $b$, and $c$ lie in the same cyclic element. Then $E(a, c) \cap E(a, b)$ is the intersection of two compact totally ordered sets whose orderings agree where possible, i.e. on the intersection, and hence has a largest element $d$. If $d$ separates $b$ and $c$, then $d$ is the required cyclic element. If $d$ does not, then $C(d, b) \cap C(d, c)$ is a true cyclic element $D$ of $X$ separating each of $b$ and $c$ from $a$. Moreover the maximality of $d$ in $E(a, c) \cap E(a, b)$ implies $D$ separates $b$ and $c$. This completes the proof.

Notation. Since the intersection of cyclic elements is a cyclic element, there is a smallest cyclic element which separates $0$, $a$, and $b$. Denote it by $D(a, b)$.

Lemma 4. Given three points $a$, $b$, and $c$ in $X$, and any $x \in D(a, b)$ and $y \in D(b, c)$, $D(x, c) = D(a, y)$.

Proof. It suffices to note that both $D(x, c)$ and $D(a, y)$ are the maximum cyclic element in $C(0, a) \cap C(0, b) \cap C(0, c)$.

To define the semilattice operation on $X$, we denote by $\wedge$ the operation on whichever of $K_n$ or $K'_n$ fits the context.

Given $a$ and $b$ in $X$, define

$$ab = h_{D(a,b)} (h_{D(a,b)}^{-1} (P(D(a, b), a)) \wedge h_{D(a,b)}^{-1} (P(D(a, b), b))).$$

Main Theorem. Let $X$ be a cell-cyclic Peano continuum. Then under the above operation $X$ is a semilattice.

Proof. The operation is obviously idempotent and commutative. It follows from Lemma 4 that it is associative. The proof of continuity will be by cases. First note that the operation is continuous when restricted to any cyclic element. Fix an open connected set $U$ containing $ab$. We seek open sets $V$ and $W$ such that $a \in V$, $b \in W$ and $VW \subset U$.

Case I. $a$, $b$, and $ab$ all distinct. If $D(a, b) = \{ab\}$ then the components of $a$ and $b$ in $X - \{ab\}$ are the required $V$ and $W$. If $D(a, b)$ is a true cyclic
element it is isomorphic to $K_n$ or $K'_n$. Thus there exist disjoint relatively open sets $V'$ and $W'$ of $D(a, b)$ containing $P(D(a, b), a)$ and $P(D(a, b), b)$, respectively, such that $V'W' \subset U$. Then $V = f_{D(a, b)}^{-1}(V')$ and $W = f_{D(a, b)}^{-1}(W')$ are the required sets.

Case II. $a = b = ab$. If this point is in $D^c$, we are done since $D$ is isomorphic to $K_n$ or $K'_n$, so suppose it is on the boundary of $D$. Choose $\epsilon > 0$ such that $B(a, \epsilon) \subset U$. The components of $X - \{a\}$ form a null sequence at most, so let $C_1, \ldots, C_n$ be those that meet $X - B(a, \epsilon/2)$. Some of these may contain a cyclic element $D_i$ containing $a$. For each of those that do, there exists a positive $\delta_i < \epsilon$ such that

$$[B(a, \delta_i) \cap D_i][B(a, \delta_i) \cap D_i] \subset B(a, \epsilon) \cap D_i$$

since $D_i$ is a topological semilattice. Also by Lemma 2, there exists a positive $\delta''_i < \epsilon_i$ such that for $x \in C_i - D_i$, both $P(D_i, x)$ in $B(a, \delta''_i)$ implies $x \in B(a, \delta''_i)$ and $x \in B(a, \delta''_i)$ implies $P(D_i, x) \in B(a, \delta''_i)$.

Then choose $\delta''_i < \delta_i$ so that $x \in (C_i - D_i) \cap B(a, \delta''_i)$ implies $P(D_i, x) \in B(a, \delta''_i)$. If $y, z \in B(a, \delta''_i) \cap C_i$, either (i) $y$ and $z$ are in the same component of $C_i - D_i$, in which case the choice of $\delta''_i$ implies $yz \in B(a, \delta''_i)$ since $P(D_i, yz) = P(D_i, y) \in B(a, \delta''_i)$, or (ii) $yz = P(D_i, y)P(D_i, z)$, which by the second implication is a product of two elements of $B(a, \delta_i) \cap D_i$, and hence in $B(a, \epsilon) \cap D_i$. Thus

$$[B(a, \delta''_i) \cap C_i][B(a, \delta''_i) \cap C_i] \subset B(a, \epsilon) \cap C_i.$$ 

Now for those $C_i$ not containing such a $D_i$, each must contain a cut point $x_i$ of itself in $B(a, \epsilon/2)$ such that $C(a, x_i) \subset B(a, \epsilon/2)$. Again by Lemma 2, there is a positive $\delta_i < \epsilon/2$ such that for $x$ in $C_i - C(a, x_i)$, $P(C(a, x_i), x) \in B(a, \delta_i)$ implies $x \in B(a, \epsilon/2)$ and $x \in B(a, \delta_i)$ implies $P(C(a, x_i), x) \in B(a, \epsilon/2)$. Choose $\delta'_i, 0 < \delta'_i < \delta_i$, so that $x \in B(a, \delta'_i) \cap [C_i \cap C(a, x_i)]$ implies $P(C(a, x_i), x) \in B(a, \delta_i)$. By an argument similar to that of the preceding paragraph, one shows that

$$[B(a, \delta'_i) \cap C_i][B(a, \delta'_i) \cap C_i] \subset B(a, \epsilon) \cap C_i.$$ 

Finally by a technique like that above choose $\delta_0$ so that $y \in B(a, \delta_0)$ and in the component of 0 in $X - \{a\}$ implies $ay \in B(a, \epsilon)$. Let $\delta$ be the smallest of all the $\delta_0, \delta'_i, \delta''_i$, and $\delta_i$. We claim that $B(a, \delta)^2 \subset B(a, \epsilon)$. For if $y, z \in B(a, \delta)$ are in the same component of $X - \{a\}$, the above two paragraphs show that $yz \in B(a, \epsilon)$. If $y$ and $z$ are in different components of $X - \{a\}$ and neither of these components contain 0, then $yz = a$, while if
the component of, say, $y$ contains 0, then $yz = ya \in B(a, \epsilon)$.

**Case III.** $a = ab \neq b$. This will be subdivided into three subcases:

(i) Suppose $D(a, b) = a$, so that $a$ separates $b$ from 0, and also that $C(a, b)$ contains a true cyclic element $D$ containing $a$. Then $P(D, b) \neq a$ and $ab = aP(D, b)$, so there exist disjoint relatively open subsets $V'$ and $W'$ of $D$ such that $a \in V'$, $P(D, b) \in W'$, and $V'W' \subset U \cap D$. By a technique similar to that of Case II, choose a $\delta > 0$ such that if $y$ is in $B(a, \delta)$ and the component of 0 in $X - \{a\}$ then $ya \in U$. Let $V = f^{-1}_D(V') \cap B(a, \delta)$ and $W = f^{-1}_D(W')$. Then for $y \in V$ and $z \in W$, either $y$ is in the component of 0 in $X - \{a\}$, in which case $yz = ya \in U$, or $y$ is not in the component of 0 in $X - \{a\}$, in which case $yz = P(D, y) \cdot P(D, z) \in V'W' \subset U$.

(ii) Suppose $D(a, b) = a$, but $C(a, b)$ contains no true cyclic element containing $a$. By Case II, there is an open set $W'$ such that $W'W' \subset U$. Also $C(a, b)$ contains a cut point $x$ with the property that $C(a, x) \subset W'$. Let $V$ be the component of $b$ in $X - \{x\}$, and let $W$ be $W'$ intersected with the component of $a$ in $X - \{x\}$. For $y \in V$ and $z \in W$, $yz = xz \in W'W' \subset U$.

(iii) Suppose $D(a, b) \neq a$. Then $D(a, b)$ is a true cyclic element containing $a$ which separates any two of $a$, $b$, and 0, so that $P(D(a, b), 0) \neq a$ and $P(D(a, b), b) \neq a$. Choose $V'$ and $W'$ disjoint relatively open subsets of $D(a, b)$ such that $P(D(a, b), b) \in W'$, $a \in V'$, $V'W' \subset U \cup D(a, b)$, but neither contains $P(D(a, b), 0)$. Then $V = f^{-1}(V')$ and $W = f^{-1}(W')$ are the required sets, for $y \in V$ and $z \in W$ imply

$$yz = P(D(a, b), y)P(D(a, b), z) \in V'W' \subset U.$$ 

This completes the proof of the main theorem.

**Corollary.** Every retract of $I^2$ admits a semilattice.

**Proof.** Borsuk [2] has characterized retracts of $I^2$ as locally connected continua which do not separate the plane. Whyburn [8] in turn has characterized these as those locally connected continua in the plane such that every true cyclic element is a simple closed curve with interior, i.e. a two-cell. From these results the Corollary follows.

**BIBLIOGRAPHY**


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