CHARACTERISTIC SUBMANIFOLDS
OF FIXED POINT FREE INVOLUTIONS

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Abstract. According to López de Medrano, a manifold with a free involution is determined by its Browder-Livesay index and the normal cobordism class of its orbit space. Here we study the Browder-Livesay indices of some cobordant involutions, namely, the characteristic submanifolds of a fixed involution.

In [2] and [3], studying fixed point free involutions on homotopy spheres, W. Browder and G. R. Livesay defined an index $\sigma$ measuring the obstruction to finding an invariant embedded sphere of codimension 1. For a closed orientable manifold $M^{4k-1}$ with a fixed point free involution $T$ which preserves orientation, the Browder-Livesay index $\sigma(T, M)$ is defined as the signature of a bilinear form on the homology of a characteristic submanifold; a characteristic submanifold $W$ is a closed, orientable submanifold of codimension 1 whose complement has two components $A$ and $B$ such that $B = TA$ and $\text{Cl } A \cap \text{Cl } B = W$. Since characteristic submanifolds arise as $f^{-1}\mathbb{R}P^n$ where $f : M/T \to \mathbb{R}P^n$ (real projective space) is the classifying map of the $Z_2$ bundle $M \to M/T$, characteristic submanifolds exist and any two are equivariantly cobordant in $M \times I$. It follows that $\sigma$ does not depend on choice of $W$ [1].

However, if $M_0$ and $M_1$ are characteristic submanifolds of $(T, X^{4k})$, in general $\sigma M_0 \neq \sigma M_1$; if $X$ is $S^{4k}$, $\sigma M_0 - \sigma M_1 = 2\sigma(T \times 1; S \times I, M_0 - M_1)$ [1] and if $M_0$ and $M_1$ are spheres, $(T, M_0)$ and $(T, M_1)$ are equivariantly diffeomorphic iff $\sigma M_0 = \sigma M_1$ [2], [3].

Here we derive a formula for $\rho(M_0, M_1) = \sigma M_0 - \sigma M_1 - 2\sigma(X \times I, M_0 - M_1)$ and show that it lies between $\pm \dim H^{2k}A$ and does assume all these values. We also give an algebraic condition for a subspace of $H^{2k}X$ to be $H^{2k}A$ where $A \cap TA$ is a characteristic submanifold of $X$. 

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Throughout this paper, all manifolds are smooth or PL, compact and orientable, all involutions are smooth or PL and have no fixed points. Homology groups have rational coefficients, except where otherwise stated. $B$ will be used for any bilinear form on a subspace of $H_{2k}N^{4k}$ given by $B(x, y) = x \cdot Ty$, the intersection number of $x$ and $Ty$ in $N$; the annihilator, under $B$, of a subspace $C$ will be denoted $\mathcal{A}C$.

Suppose $S$ is an orientation reversing free involution on a compact manifold $Y^{4k+1}$:

Definition. A characteristic submanifold $N^{4k}$ of $(S, Y)$ is an invariant submanifold of $Y$, meeting $\partial Y$ transversely (in $\partial N$) for which there are compact submanifolds $U$ and $V$ of $Y$ with $U \cup V = Y$, $U \cap V = N$ and $V = SU$. Then $\partial N$ will be a characteristic submanifold of $(S, \partial Y)$ splitting $\partial Y$ into $(U \cap \partial Y) \cup (V \cap \partial Y)$.

Definition. The Browder-Livesay index $\sigma(S; Y, N)$ is the signature of the symmetric bilinear form $B(x, y) = x \cdot Sy$ defined on $\ker(H_{2k}N \to H_{2k}U)$. In fact, given $\partial N$ this is independent of $N$.

Now suppose $(T, M^{4k-1})$ is a given characteristic submanifold of $(T, X^{4k}) = (S|\partial Y, \partial Y)$. Then $M$ can be extended to a characteristic submanifold $N^{4k}$ of $Y$, splitting $Y$ into $U \cup V$; let $A = U \cap X$ and $B = V \cap X$. Notice that $T/X$ reverses and $T/M$ preserves the orientation.

Suppose all maps in the diagram are induced by the inclusions. Let $K = u^{-1}\operatorname{im} i \cap v^{-1}\operatorname{im} j \subseteq H_{2k}N$, and $L = \operatorname{im} a \cap \operatorname{im} b \subseteq H_{2k}Y$.

![Diagram](attachment:image.png)

Define a bilinear form $\rho$ on $L$ as follows: if $x_i = a(s_i) = b(t_i)$, $i = 1, 2$, set $\rho(x_1, x_2) = s_1 \cdot Tt_2$, where the intersection is taken in $X$.

**Lemma.** $\rho$ is well defined (i.e. is independent of the choices of $s_i$ and $t_i$) and is induced from the bilinear form $-B$ on $K$.

**Proof.** First we show there is an induced form. By the Mayer-Vietoris
characteristic submanifolds of fixed involutions

sequence for \( Y \), the sequence

\[
0 \rightarrow \ker \hat{u} \cap K \rightarrow K \rightarrow L \rightarrow 0
\]

is exact. The kernel of \( -B \) restricted to \( K \) is \( \ker B \cap K \), where

\[
\ker B = \ker(u^{-1} i \cap v^{-1} j) = \ker(u^{-1} i) + \ker(v^{-1} j) \supset \ker v + \ker u,
\]

since \( \ker v = u^{-1} i \). But by the Mayer-Vietoris sequence for \( Y \), \( \ker u + \ker v = \ker ku \); therefore \( \ker ku \cap K \subseteq \ker -B | K \), so that \( -B \) induces a bilinear form on \( L \).

To show that this induced form is \( \rho \), let \( x_1 \) and \( x_2 \in L \), and pick \( s_i \) and \( t_i \); \( i = 1, 2 \), such that \( x_i = as_i = bt_i \); then there exist \( m_i \in K \) such that

\[
um_i = ts_i \quad \text{and} \quad vm_i = it_i.
\]

If \( -s_i \cdot Tt_2 = m_1 \cdot Sm_2 \), since \( x_i = kum_i \), then \( \rho \) is the induced form. In fact, since \( \partial U = A \cup N \), intersection numbers are equal whether taken in \( A \) and \( N \) or in \( \partial U \). Regarded as elements of \( H_{2k} \partial U, (s_1 - m_1) \) and \( T(t_2 - m_2) \in \ker(H_{2k} \partial U \rightarrow H_{2k} U) \), hence their intersection number is 0. Since \( H_{2k} A \) and \( H_{2k} N \) are orthogonal in \( H_{2k} \partial U \), \( s_1 \cdot Tt_2 + m_1 \cdot Sm_2 = 0 \) i.e. \( s_1 \cdot Tt_2 = -m_1 \cdot Sm_2 \). Therefore \( \rho \) is the induced form and hence is well defined.

Theorem 1. \( \sigma(T, M) = 2\sigma(S; Y, N) - \text{signature} \rho \).

Proof. Since \( (T, M) = \partial(S, N) \) and \( S \) has no fixed points, \( \sigma(T, M) = \text{signature} B \) on \( H_{2k} N \), where \( B(x, y) = x \cdot Sy \) [4], [5].

Notice that if \( F \) is a bilinear form on a vector space \( W \), and if \( C \) is a subspace for which \( C \subseteq \ker F \), then signature \( F = \text{signature} F|_{\ker F} \). For by replacing \( C \) by \( C + \ker F \), we may assume \( \ker F \subseteq C \); then \( W \) can be written as \( C \oplus D \oplus E \) where \( \ker F = C \oplus D \) and \( \ker E = C \oplus E \). Now \( \text{sig} F = \text{sig} F|_{C \oplus E} + \text{sig} F|_D \), but since the annihilator of \( C \) in \( C \oplus E \) is \( C \), \( \text{sig} F|_{C \oplus E} = 0 \), and since \( C \subseteq \ker F|_D \oplus C \), \( \text{sig} F = \text{sig} F|_{\ker F} \).

By Poincaré duality, the kernel of \( B \) on \( H_{2k} N \) is \( \ker p \) where \( p : H_{2k} N \rightarrow H_{2k} N \), \( M \) is induced by the projection. Let \( C = (\ker u + \ker p) \cap \ker u \), then since \( C \subseteq \ker F \), \( \text{sig} B = \text{sig} B|_{\ker F} \); but \( \ker F = \ker u + \ker u \) where these two terms are orthogonal, so that

\[
\text{sig} B = \text{sig} B|_{\ker u} + \text{sig} B|_{\ker u}.
\]

Let \( W = \ker u \) and let \( J = \ker B|_W \). Consider \( D = \ker v \cap W \) and \( E = \ker v + J \), where \( E \subseteq W \) since \( \ker v \subseteq \ker u \). Then \( \text{sign} E \) (in \( W \)) is \( \ker v \cap \ker J \cap W = D \) and since \( E \supseteq J \), \( E = \ker B \cap D \); so that \( \text{sign} (E \cap D) = \text{sign} D \).
Therefore $\operatorname{sign} B|W = \operatorname{sign} B|\partial (E \cap D) = \operatorname{sign} B|E \oplus D$, which is equal, since $E$ and $D$ are orthogonal, to $\operatorname{sign} B|E + \operatorname{sign} B|D$.

Since $J \subseteq \ker B|E$, $\operatorname{sign} B|E = \operatorname{sign} B|\ker v$, and $S$, preserving orientation on $N$, also preserves intersection numbers, so that $\operatorname{sign} B|E = \operatorname{sign} B|\ker u$.

Finally, $D = \partial \ker v \cap \partial \ker u = u^{-1}im i \cap v^{-1}im j = K$; since $\rho$ is induced from $-B|K$, by dividing out by part of the kernel, $B|D = \operatorname{sign} -p$. Therefore

$$\sigma(T, M) = 2 \operatorname{sign} B|\ker u - \operatorname{sign} \rho = 2 \sigma(S; Y, N) - \operatorname{sign} \rho.$$  

Note. Since $\sigma(T, M)$ and signature $\rho$ are determined by the embedding of $\partial N$ in $\partial Y$, $\sigma(S; Y, N)$ does not depend on $N$, but only on $\partial N$, and will be denoted $\sigma(S; Y, \partial N)$.

Corollary. Suppose $M_0$ and $M_1$ are two characteristic submanifolds of $(T, X^{4k})$ where $T$ reverses orientation and $X$ is closed. Then

$$\sigma(M_0) - \sigma(M_1) = 2\sigma(T \times 1; X \times 1, M_0 - M_1) - \operatorname{sign} \rho.$$  

In particular, when $H_{2k}X = 0$, $\sigma(M_0) - \sigma(M_1) = 2\sigma(T \times 1; X \times 1, M_0 - M_1)$ [1] and when $X$ is a homotopy sphere, this is a multiple of 16 since the bilinear forms are even and unimodular. However, in general $\sigma(M_0) - \sigma(M_1)$ need not be even; here is an example where $\sigma(M_0) = 0$ and $\sigma(M_1) = 1$.

Let $X = S^{2k} \times S^{2k}$ and $T = A \times 1$ where $A$ is the antipodal map.

Let $M_0 = S^{2k-1} \times S^{2k}$, a characteristic submanifold of $M_0$ is $W_0 = S^{2k-2} \times S^{2k}$ and since $H_{2k-1}W_0 = 0$, $\sigma(M_0) = 0$.

Let $M_1 = \{(x, y) \in S^{2k} \times S^{2k} | x \cdot y = 0\}$. Then $\rho: M_1 \to S^{2k}$ (projection on the first factor) is a fibre bundle with fibre $S^{2k-1}$. $S^{2k}$ has a characteristic submanifold $S$ dividing it into two discs, $D^+$ and $D^-$. Then $W_1 = p^{-1}S$ is a characteristic submanifold of $M_1$, dividing it into $p^{-1}D^+ \cup p^{-1}D^-$. Since the bundle is trivial when restricted to $S$, $H_{2k-1}W_1 = Q \oplus Q$, where one generator $F$ is represented by the fibre, and the other, $C$, is represented by the cross-section $S^{2k-1} \to M_1, x \to (x, \alpha(x))$ where $\alpha(0, x_1, \ldots, x_{2k}) = (0, -x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$. Now $TC = C$ since $\alpha \sim \Lambda \alpha$, and $TF = -F$. $p^{-1}D^+ = D^+ \times S^{2k-1}$ and the map $H_{2k-1}W_1 \to H_{2k-1}p^{-1}D^+$ is determined by $C \to F$ and $F \to F$. Therefore ker $H_{2k-1}W_1 \to H_{2k-1}p^{-1}D^+ = Q$ and a generator is $C - F$. Since

$$\sigma(M_1) = 1.$$  

In this example $L = Q \oplus Q = H_{2k}\partial(X \times 1)$ and $\sigma \rho = 1; \sigma(T \times 1) = 0$.  

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Proposition. If $T$ is an orientation reversing involution on a manifold $X^{4k}$ with boundary, then given $(M_0, \partial M_0)$ a characteristic submanifold of $(X, \partial X)$ where $\partial M_0$ divides $\partial X$ into $P \cup TP$, then there is a characteristic submanifold $(M_1, \partial M_1)$ with $\alpha M_1 = \alpha M_0$, such that $\partial M_1 = \partial M_0$, $H_r(X, A_1; \mathbb{Z}) = 0$ for $r < 2k$ and $H_r(A_1, P; Q) \to H_r(X, P; Q)$ is a monomorphism for $r \leq 2k$.

The proof is by equivariant handle exchange between $A$ and $B$ to make the pair $(A, M)$ $2k-1$ connected.

Suppose inductively that $H_p(A_0, M_0; \mathbb{Z}) = 0$ for all $p < \text{some fixed } r \leq 2k$. Then the Hurewicz map $\pi_r(A_0, M_0) \to H_r(A_0, M_0; \mathbb{Z})$ is surjective. Let \{x_1, \ldots, x_n\} $\subset \pi_r(A_0, M_0)$ represent a basis for $H_r(A_0, M_0; \mathbb{Z})$. If $r < 2k$, $x_1$ can be represented by an embedding $D^{r-1} \times S^{4k-r-1} \subset A_0, M_0$ which is disjoint from $T x_1$ [8]. Let $H$ be a tubular neighborhood of $D^r, S^{r-1}$ in $A_0, M_0$, so that $H = D^r \times D^{4k-r}$ and $H \cap M_0 = S^{r-1} \times D^{4k-r}$, which is disjoint from $TH$; and let $A_1, B_1$ and $M_1$ be the manifolds resulting from exchanging $H$ and $TH$, that is $A_1 = A_0 - H \cup TH$ and $B_1 = B_0 \cup H - TH$ and

$$M_1 = M_0 - (S^{r-1} \times D^{4k-r}) \cup (D^r \times S^{4k-r-1})$$

$$- T(S^{r-1} \times D^{4k-r}) \cup T(D^r \times S^{4k-r-1}).$$

Now $(A_0, M_0)$ is equivariantly cobordant to $(A_1, M_1)$ in $X \times I$; in particular let $K$ be a tubular neighborhood of $(D^r, S^{r-1})$ in $(A_0 \times I, M_0 \times 1)$, and set

$$U = A_0 \times I - K \cup TK,$$

$$V = (T \times 1) U = B_0 \times I \cup K - TK \text{ and } N = U \cap V.$$

Then clearly $N$ is a characteristic submanifold of $X \times I$ and $(U, N)$ is a cobordism between $(A_0, M_0)$ and $(A_1, M_1)$.

Then $H_r(A_1, M_1; \mathbb{Z})$ is generated by \{x_2, \ldots, x_n\} and $H_*(A_1, M_1; \mathbb{Z}) = H_*(A_0, M_0; \mathbb{Z})$ in dimensions $< r$. Since $N$ is the trace of surgery on $x_1$ and $Tx_1, H_{2k}(N, M_0; \mathbb{Z}) = 0$ and $H_{2k} M_0 \to H_{2k} N$ is an epimorphism. Since $M_0 \subset \partial N$ and intersection numbers are 0 on boundaries, $\alpha M_0 - \alpha M_1 = \text{sign } B|H_{2k} N [4, 5] = 0$.

When $r = 2k$, $x_1$ can still be represented by an embedding of $D^{2k}, S^{2k-1}$ into $A_0, M_0 [7, p. 39]$ disjointly from $Tx_1$. If $x_1$ has infinite order in $H_{2k}(A_0, M_0; \mathbb{Z})$, surgery on $x_1$ will reduce the rank of $H_{2k}(A_0, M_0; Q)$ if and only if $\partial Tx_1 \neq 0$ in $H_{2k-1}(A_0, P; Q)$. In this case $H_{2k} M_0$ is isomorphic to $H_{2k} N$, because in the exact sequence

$$0 \to H_{2k} M_0 \to H_{2k} N \to O \oplus OT^* \xrightarrow{\partial} H_{2k-1} M_0,$$
\[ \partial \text{ is a monomorphism since the composition } \]
\[ Qx \oplus QT_x \to H_{2k-1}M_0 \to H_{2k-1}A_0, \quad P \oplus H_{2k-1}B_0, \quad TP \]
\[ \text{ sends } (ax, bT_x) \text{ into } (b\partial T_x, a\partial x). \text{ Therefore, as before, } \sigma M_0 = \sigma M_1. \]

We can assume, then, that \( M \) is a characteristic submanifold with \( H_*(A, M; \mathbb{Z}) = 0 \) in dimensions \(< 2k\), and for which the composition

\[
H_{2k}A, M \to H_{2k-1}M \xrightarrow{T} H_{2k-1}M \to H_{2k-1}A, P
\]
is 0, that is, since \( T \) is an isomorphism, that the maps

\[
H_{2k}A, M \to H_{2k-1}B, TP \quad \text{and} \quad H_{2k}B, M \to H_{2k-1}A, P
\]
are 0. It follows that in the Mayer-Vietoris sequence

\[ H_{2k}X, P \to H_{2k-1}M \to H_{2k-1}A, P \oplus H_{2k-1}B \to H_{2k-1}X, P, \]

\[ \text{im}(H_{2k}X, P \to H_{2k-1}M) = \ker(H_{2k-1}M \to H_{2k-1}B). \]

Taking duals we see that \( \text{im } i = \text{im } j \)

\[
H^{2k}X, TP \xrightarrow{i} H^{2k}B, TP \xrightarrow{j} H^{2k+1}X, B \xrightarrow{\cong} H^{2k+1}A, M
\]

so that \( h \) is an epimorphism, from which it follows that \( H_{2k}(A, P) \to H_{2k}(X, P) \) is a monomorphism. From the 5 lemma \( H_*(A, P) \to H_*(X, P) \) is a monomorphism in all dimensions \(< 2k\).

**Theorem 2.** Let \( T \) be an orientation reversing involution of \( X^{4k} \) where \( X \) is a simply connected closed manifold and \( k > 1 \). Let \( C \) be a (vector) subspace of \( H_{2k}(X; \mathbb{Q}) \). Then there is a characteristic submanifold \( M \) dividing \( X \) into \( A \cup B \) with \( H_{2k}A = C \), if and only if \( C = \partial C \), that is, since \( B \) is nonsingular, if and only if \( C \cdot TC = 0 \) and \( \dim C = \frac{1}{2} \dim H_{2k}X \).

**Proof.** \( \Leftarrow \) Clearly if \( C = H_{2k}A, C \cdot TC = 0 \). Since the Alexander dual of \( \text{im}(H_{2k}A \to H_{2k}X) \) is \( \text{im}(H^{2k}X, B \to H^{2k}X) \) which is isomorphic to \( \text{im}(H_{2k}X \to H_{2k}X, A) \), and since by exactness,

\[
\dim \text{im}(H_{2k}A \to H_{2k}X) + \dim \text{im}(H_{2k}X \to H_{2k}X, A) = \dim H_{2k}X,
\]

\[ 2 \dim C = \dim H_{2k}X. \]

\( \Rightarrow \) Suppose \( f \) is an immersion: \( M^{2k} \to X \) with \( f_*[M] = x \in H_{2k}X \). Then there is an immersion \( g \) of \( M \to M \), approximating \( f \), with \( g \) disjoint from
$T_g$, because if $p$ is the projection: $X \to X/T$, then the self-intersection number of $p_*2x$ will have no component in the generator of $\pi_1X/T$, so that $p_*2x$ can be represented without any intersections arising from intersections of $x$ and $Tx$ [7, p. 45]. We can then represent a basis for $C$ by immersed manifolds $M_i$, each disjoint from all $TM_j$ [6]. Since $X$ is 1-connected, the image of each $M_i$ can be assumed 1-connected. Let $U$ be a regular neighborhood of the union of the $M_i$ disjoint from $TU$. $X - U - TU$ has a closed characteristic submanifold $M$ dividing it into $Y \cup TY$, where $\partial Y = \partial U$, which can be chosen so that $H_r(Y, M = 0$, $r < 2k$, and $H_r(Y, \partial U) \to H_r(Y \cup TY, \partial U)$ is a monomorphism for $r \leq 2k$. Set $A = U \cup Y$.

Now image $H_{2k}A \to H_{2k}X = C$ because image $H_{2k}A \to H_{2k}X$ is contained in its annihilator under $B$ and contains $C$ which equals $\mathcal{G}C$. Therefore it remains to show that $H_{2k}A \to H_{2k}X$ is a monomorphism.

Now $a$ is a monomorphism, hence so are $b$ and $c$. Thus $\ker b \subset \ker c$; but since $g$ is a monomorphism, $\ker f \subset \ker e$; therefore $\ker d = 0$. $h$ is a monomorphism because

$$
\ker h = \text{im}(H_{2k+1}X, X - TU \to H_{2k}X - TU)
$$

and $H_{2k+1}X, X - TU = H^{2k-1}TU$ which is 0 since each $H_1M = 0$. Therefore $hd$ is a monomorphism: $H_{2k}A \to H_{2k}X$.

As an application we have
Theorem 3. Let $T$ be an orientation reversing involution of $X^{4k}$ where $X$ is a simply connected closed manifold and $k > 1$. Then there is a characteristic submanifold $M_0$ of $X$ such that given $|r| \leq n = \dim H_{2k}X$, there is another characteristic submanifold of $M_1$ of $X$ for which

$$\sigma(\rho(M_0, M_1)) = \sigma(M_0) - \sigma(M_1) - 2\sigma(X \times T_1 M_0 - M_1) = r.$$

Proof. First we find a subspace $C$ of $H_{2k}X$ for which $C \oplus TC = H_{2k}X$ and $C \cdot TC = 0$, then $C$ can be realized as $H_{2k}(A)$ (Theorem 2); set $M_0 = \partial A$. Let $V = \{x : x = Tx\}$ and $W = \{x : x = -Tx\} \subset H_{2k}X$, then $H_{2k}X = V \oplus W$, but since $T$ has Lefschetz number 0 and reverses orientation, trace $T : H_{2k}X \rightarrow H_{2k}X = 0$, so that $\dim Y = \dim W = \frac{1}{2} \dim H_{2k}X$, and $V = \bar{V}$ and $W = \bar{W}$. Therefore $H_{2k}X$ has a symplectic basis $\{v_i, w_i : i = 1, \ldots, n/2\}$, that is, $v_i \cdot v_j = w_i \cdot w_j$ for all $i$ and $j$ and $v_i \cdot w_j \neq 0$ only if $i = j$ and $v_i \cdot w_i = 1$, with $Tv_i = v_i$ and $Tw_i = -w_i$. Let $C$ be the space generated by $\{v_i + w_i : i = 1, \ldots, n/2\}$; then $C \cap TC = 0$, $C + TC = H_{2k}X$ and $C \cdot TC = 0$.

Given $n/2$ pairs of rational numbers $(\alpha_i, \beta_i)$ with each $\alpha_i < \beta_i$, let $C_1$ be the subspace of $H_{2k}X$ generated by $\{\alpha_i v_i + \beta_i w_i\}$; like $C$, $C_1$ can be realized by $A_i$; let $M_1 = \partial A_i$. We wish to choose $(\alpha_i, \beta_i)$ so that $\sigma(\rho(M_0, M_1)) = r$.

Now $\rho$ is defined on $(C + C_1) \cap T(C + C_1)$ which is $H_{2k}X$. Let $X_i$ denote the subspace of $H_{2k}X$ generated by $v_i$ and $w_i$; then the family of $X_i$'s are mutually orthogonal under $B$, hence also under $\rho$, since

$$\rho(c + c_1, c' + c_1') = B(c, c') - B(c_1, c_1').$$

Consequently $\sigma(\rho) = \Sigma \sigma(\rho|X_i)$. But

$$\rho|X_i = \begin{pmatrix} 2\beta_i/(\beta_i - \alpha_i) & 0 \\ 0 & 2\alpha_i/(\beta_i - \alpha_i) \end{pmatrix};$$

since $\alpha_i < \beta_i$, $\sigma(\rho|X_i) = \sigma(\beta_i) + \sigma(\alpha_i)$.

Clearly then $\alpha_i$ and $\beta_i$ can be chosen so that $\sigma(\rho|X_i)$ is $2, 1, 0$, or $-2$. It follows that $\sigma(\rho)$ can be any number between $-n$ and $n$.

Since the maximum dimension, for all $M_0$ and $M_1$ of the space $L$ on which $\rho$ is defined, is $n$, $\rho(M_0, M_1)$ can assume all possible values.

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