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SOLUTION OF INTEGRAL EQUATIONS BY PRODUCT INTEGRATION

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ABSTRACT. Suppose f, h and G are functions with values in a normed complete ring. With suitable restrictions on these functions, it is established that

$$f(x) = h(x) + \int_a^x f(u)G(u, v)$$

for $a \le x \le b$ only if $\int_a^x h(u)G(u, v)\,{}_v\Pi^x (1 + G)$ exists and is $f(x) - h(x)$ for $a \le x \le b$, and that

$$f(x) = h(x) + \int_a^x G(u, v)f(u)$$

for $a \le x \le b$ only if $\int_a^x {}_x\Pi^v (1 + \mathcal{G})G(u, v)h(u)$ exists and is $f(x) - h(x)$ for $a \le x \le b$, where $\mathcal{G}(s, r) = G(r, s)$.

Lower case letters are used to represent functions from R to N, and capital letters are used to represent functions from $R \times R$ to N, where R denotes the set of real numbers and N denotes a ring which has a multiplicative identity element represented by 1 and a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. For a subdivision $\{x_i\}_{i=0}^n$ of an interval $[a, b]$, we use G_i and f_i to denote $G(x_{i-1}, x_i)$ and $f(x_i)$, respectively.

The statement that $G \in OB^\circ$ on $[a, b]$ means there exist a subdivision D of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then $\Sigma_{i=1}^n |G_i| < B$.

The statement that $\int_a^b G$ exists means there exists an element L of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then $|L - \Sigma_{i=1}^n G_i| < \epsilon$. Further, $G \in OA^\circ$ on $[a, b]$ only if $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$.

The statement that ${}_a\Pi^b (1 + G)$ exists means there exists an element L

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of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then $|L - \Pi_{i=1}^n (1 + G_i)| < \epsilon$. Further, $G \in OM^\circ$ on $[a, b]$ only if ${}_x\Pi^y(1 + G)$ exists for $a \leq x \leq y \leq b$ and $\int_a^b |1+G-\Pi(1+G)| = 0$. The existence of integrals is defined similarly on intervals $[p, q]$, where $q < p$.

The function h is quasi-continuous on $[a, b]$ only if $\lim_{x \to p-} h(x)$ exists for $a < p \leq b$ and $\lim_{x \to p+} h(x)$ exists for $a \leq p < b$. Further, $G \in OL^\circ$ on $[a, b]$ only if $\lim_{x \to p-} G(x, p)$ and $\lim_{x,y \to p-} G(x, y)$ exist for $a < p \leq b$, and $\lim_{x \to p+} G(p, x)$ and $\lim_{x,y \to p+} G(x, y)$ exist for $a \leq p < b$. See B. W. Helton [2] and J. S. Mac Nerney [6] for additional background.

The theorems in this paper were suggested by a result of J. S. Mac Nerney [5, Theorem 3.4, p. 361]. However, their proofs are constructed by using techniques similar to those employed by B. W. Helton [2, §5, pp. 307–314]. There, the integral equations

$$f(x) = h(x) + \int_a^x f(u)G(u, v) \quad \text{and} \quad f(x) = h(x) + \int_a^x G(u, v)f(u)$$

were solved with the restriction that h have bounded variation. We now solve these integral equations with the restriction that h be quasi-continuous. The reader is also referred to related results by D. B. Hinton [4, Theorems 4.1, 4.2, 4.3, 4.4, pp. 325–327] and C. W. Bitzer [1, Theorems 4.1, 4.2, 5.9, pp. 447–451].

We now state three lemmas that are used in the development of the results in this paper.

Lemma 1. *If G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a, b]$, then the following statements are equivalent:*

(1) *$G \in OA^\circ$ on $[a, b]$, and*

(2) *$G \in OM^\circ$ on $[a, b]$ [2, Theorem 3.4, p. 301].*

Lemma 2. *If H and G are functions from $R \times R$ to N, $H \in OL^\circ$ on $[a, b]$ and G is in OA° and OB° on $[a, b]$, then GH and HG are in OA° and OM° on $[a, b]$ [3, Theorem 2, p. 494].*

Lemma 3. *If F and G are functions from $R \times R$ to N belonging to OB° on $[a, b]$, $F \in OA^\circ$ on $[a, b]$ and each of ${}_x\Pi^y(1 + G)$ and $\int_x^y F(u, v)\, {}_v\Pi^y(1 + G)$ exists for $a \leq x < y \leq b$, then*

$$\int_a^b \left| F(x, y) - \int_x^y F(u, v)\, {}_v\prod^y (1 + G) \right|$$

exists and is zero [2, Lemma, p. 307].

Theorem 1. *If f and h are functions from R to N, G is a function from $R \times R$ to N, h is quasi-continuous on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, then the following statements are equivalent:*

(1) *f is bounded on $[a, b]$, $G \in OA^\circ$ on $[a, b]$, $f(u)G(u, v) \in OA^\circ$ on $[a, b]$ and $f(x) = h(x) + \int_a^x f(u)G(u, v)$ for $a \leq x \leq b$, and*

(2) *$G \in OM^\circ$ on $[a, b]$ and $\int_a^x h(u)G(u, v)\,{}_v\Pi^x(1 + G)$ exists and is $f(x) - h(x)$ for $a \leq x \leq b$.*

Proof. (1) $\rightarrow$ (2). Since G is in OA° and OB° on $[a, b]$, it follows from Lemma 1 that $G \in OM^\circ$ on $[a, b]$. Suppose $a \leq x \leq b$. If $a = x$, then (2) follows immediately. Therefore, suppose $a < x$. The existence of $I(a, x)$ follows from Lemma 2, where

$$I(r, s) = \int_r^s h(u)G(u, v)\,{}_v\prod^s (1 + G)$$

for $a \leq r \leq s \leq b$. We now show that $I(a, x)$ is equal to $f(x) - h(x)$. Let $\epsilon > 0$.

There exists a subdivision D_1 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

$$\left| I(a, x) - \sum_{i=1}^n h_{i-1}G_i\,{}_{x_i}\prod^x (1 + G) \right| < \epsilon/3.$$

There exist a subdivision D_2 of $[a, x]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then

(1) $\Pi_{i=1}^n (1 + |G_i|) < B$, and

(2) $\Sigma_{i=1}^n |h_{i-1}G_i| < B$.

Since G is in OM° and OB° on $[a, x]$, there exists a subdivision D_3 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_3 and $1 \leq i \leq n$, then

$$\left| {}_{x_i}\prod^x (1 + G) - \prod_{k=i+1}^n (1 + G_k) \right| < \epsilon(3B)^{-1}.$$

Since $f(u)G(u, v) \in OA^\circ$ on $[a, x]$, there exists a subdivision D_4 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_4, then

$$\sum_{i=1}^n \left| f_{i-1}G_i - \int_{x_{i-1}}^{x_i} f(u)G(u, v) \right| < \epsilon(3B)^{-1}.$$

Let D denote the subdivision $\bigcup_{i=1}^4 D_i$ of $[a, x]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. For $1 \leq i \leq n$, we have that

$$f_i - f_{i-1} = h_i - h_{i-1} + f_{i-1}G_i + c_i,$$

where

$$c_i = \int_{x_{i-1}}^{x_i} f(u)G(u, v) - f_{i-1}G_i.$$

Thus,

$$f_i = h_i - h_{i-1} + f_{i-1}(1 + G_i) + c_i.$$

Now, by using iteration for $i = 1, 2, \cdots, n$, we have that

$$f_n = h_n + \sum_{i=1}^{n} h_{i-1}G_i \prod_{k=i+1}^{n} (1 + G_k) + \sum_{i=1}^{n} c_i \prod_{k=i+1}^{n} (1 + G_k).$$

Hence,

$$\begin{aligned}
&|I(a, x) - \{f(x) - h(x)\}| \\
&= \Bigg| h(x) + I(a, x) - h_n \\
&\qquad - \sum_{i=1}^{n} h_{i-1}G_i \prod_{k=i+1}^{n} (1 + G_k) - \sum_{i=1}^{n} c_i \prod_{k=i+1}^{n} (1 + G_k) \Bigg| \\
&\le \left| \sum_{i=1}^{n} h_{i-1}G_i \prod_{x_i}^{x} (1 + G) - \sum_{i=1}^{n} h_{i-1}G_i \prod_{k=i+1}^{n} (1 + G_k) \right| \\
&\quad + \left| I(a, x) - \sum_{i=1}^{n} h_{i-1}G_i \prod_{x_i}^{x} (1 + G) \right| + \sum_{i=1}^{n} |-c_i| \left| \prod_{k=i+1}^{n} (1 + G_k) \right| \\
&< \sum_{i=1}^{n} |h_{i-1}G_i| \left| \prod_{x_i}^{x} (1 + G) - \prod_{k=i+1}^{n} (1 + G_k) \right| + \epsilon/3 + B[\epsilon(3B)^{-1}] \\
&< B[\epsilon(3B)^{-1}] + 2\epsilon/3 = \epsilon.
\end{aligned}$$

Therefore, (1) implies (2).

Proof. (2) $\longrightarrow$ (1). Since h is bounded on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, it follows that f is bounded on $[a, b]$. Since G is in OM° and OB° on $[a, b]$, it follows from Lemma 1 that $G \in OA^\circ$ on $[a, b]$. Further, it follows by employing Lemma 2 that $f(u)G(u, v) \in OA^\circ$ on $[a, b]$. Suppose $a \le x \le b$. If $a = x$, then (2) follows immediately. Therefore, suppose $a < x$. Let $\epsilon > 0$.

There exists a subdivision D_1 of $[a, x]$ such that, if $\{x_i\}_{i=0}^{n}$ is a refinement of D_1, then

$$\left| \int_a^x f(u)G(u, v) - \sum_{i=1}^{n} f_{i-1}G_i \right| < \frac{\epsilon}{3}.$$

There exists a number B such that, if $a \leq t \leq x$, then $|I(a, t)| < B$, where $I(a, t)$ is defined as in the first part of the proof. Now, since $G \in OM^{\circ}$ on $[a, x]$, there exists a subdivision D_2 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then

$$\sum_{i=1}^{n} \left| 1 + G_i - \prod_{x_{i-1}}^{x_i} (1 + G) \right| < \epsilon(3B)^{-1}.$$

It follows by applying Lemma 3 that there exists a subdivision D_3 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_3, then

$$\sum_{i=1}^{n} |h_{i-1} G_i - I(x_{i-1}, x_i)| < \frac{\epsilon}{3}.$$

Let D denote the subdivision $\bigcup_{i=1}^{3} D_i$ of $[a, x]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Thus,

$$\begin{aligned}
&\left| h(x) + \int_a^x f(u)G(u, v) - f(x) \right| \\
&\leq \left| h(x) + \sum_{i=1}^{n} f_{i-1} G_i - h(x) - I(a, x) \right| + \left| \int_a^x f(u)G(u, v) - \sum_{i=1}^{n} f_{i-1} G_i \right| \\
&< \left| \sum_{i=1}^{n} \{h_{i-1} + I(a, x_{i-1})\} G_i - I(a, x) \right| + \epsilon/3 \\
&= \left| \sum_{i=1}^{n} \{h_{i-1} G_i + [I(a, x_{i-1})][1 + G_i] - I(a, x_{i-1})\} - I(a, x) \right| + \epsilon/3 \\
&\leq \left| \sum_{i=1}^{n} \left\{ h_{i-1} G_i + [I(a, x_{i-1})] \left[\prod_{x_{i-1}}^{x_i} (1 + G) \right] - I(a, x_{i-1}) \right\} - I(a, x) \right| \\
&\quad + \sum_{i=1}^{n} |I(a, x_{i-1})| \left| 1 + G_i - \prod_{x_{i-1}}^{x_i} (1 + G) \right| + \epsilon/3 \\
&< \left| \sum_{i=1}^{n} \{I(a, x_i) - I(a, x_{i-1})\} - I(a, x) \right| \\
&\quad + \sum_{i=1}^{n} |h_{i-1} G_i - I(x_{i-1}, x_i)| + B[\epsilon(3B)^{-1}] + \epsilon/3 \\
&< 0 + \epsilon/3 + 2\epsilon/3 = \epsilon.
\end{aligned}$$

Therefore, (2) implies (1).

Theorem 2. *If f and h are functions from R to N, G is a function from $R \times R$ to N, $\mathcal{G}(y, x) = G(x, y)$ for $a \leq x < y \leq b$, h is quasi-continuous on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, then the following statements are equivalent:*

(1) *f is bounded on $[a, b]$, $G \in OA^\circ$ on $[a, b]$, $G(u, v)f(u) \in OA^\circ$ on $[a, b]$ and $f(x) = h(x) + \int_a^x G(u, v)f(u)$ for $a \leq x \leq b$, and*

(2) *$\mathcal{G} \in OM^\circ$ on $[b, a]$ and $\int_{a\ x}^{x} \Pi^v (1 + \mathcal{G})G(u, v)h(u)$ exists and is $f(x) - h(x)$ for $a \leq x \leq b$.*

The proof of Theorem 2 is similar to the proof of Theorem 1, and therefore, we omit it.

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