ABSTRACT. Suppose \( f, h \) and \( G \) are functions with values in a normed complete ring. With suitable restrictions on these functions, it is established that

\[
f(x) = h(x) + \int_a^x f(u) G(u, v) \, \text{d}u
\]

for \( a \leq x \leq b \) only if \( \int_a^x h(u) G(u, v) \, \text{d}u \) exists and is \( f(x) - h(x) \) for \( a \leq x \leq b \), and that

\[
f(x) = h(x) + \int_a^x G(u, v) f(u) \, \text{d}u
\]

for \( a \leq x \leq b \) only if \( \int_a^x \Pi^v (1 + G) h(u) \) exists and is \( f(x) - h(x) \) for \( a \leq x \leq b \), where \( \Pi^v (s, r) = G(r, s) \).

Lower case letters are used to represent functions from \( \mathbb{R} \) to \( \mathbb{N} \), and capital letters are used to represent functions from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{N} \), where \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{N} \) denotes a ring which has a multiplicative identity element represented by 1 and a norm \(| \cdot |\) with respect to which \( \mathbb{N} \) is complete and \(|1| = 1 \). For a subdivision \( \{x_i\}_{i=0}^n \) of an interval \([a, b]\), we use \( G_i \) and \( f_i \) to denote \( G(x_{i-1}, x_i) \) and \( f(x_i) \), respectively.

The statement that \( G \in OB^\vee \) on \([a, b]\) means there exist a subdivision \( D \) of \([a, b]\) and a number \( B \) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \), then \( \sum_{i=1}^n |G_i| < B \).

The statement that \( \int_a^b G \) exists means there exists an element \( L \) of \( \mathbb{N} \) such that, if \( \epsilon > 0 \), then there exists a subdivision \( D \) of \([a, b]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \), then \( |L - \sum_{i=1}^n G_i| < \epsilon \). Further, \( G \in OA^\vee \) on \([a, b]\) only if \( \int_a^b G \) exists and \( \int_a^b |G - f G| = 0 \).

The statement that \( \int_a^b (1 + G) \) exists means there exists an element \( L \)

Presented to the Society, March 4, 1974; received by the editors January 17, 1974.

AMS (MOS) subject classifications (1970). Primary 45N05; Secondary 26A39, 26A42.

Key words and phrases. Sum integral, product integral, subdivision-refinement integral, integral equation, interval function, normed complete ring.
of \( N \) such that, if \( \epsilon > 0 \), then there exists a subdivision \( D \) of \([a, b]\) such that, if 
\[
\left| L - \Pi^n_{i=1} (1 + G_i) \right| < \epsilon.
\]
Further, \( G \in OM^o \) on \([a, b]\) only if \( \int x \Pi^y (1 + G) \) exists for \( a \leq x \leq y \leq b \) and 
\[
\int^b_a |1 + G - \Pi(1 + G)| = 0.
\]
The existence of integrals is defined similarly on intervals \([p, q]\), where 
\( q < p \).

The function \( h \) is quasi-continuous on \([a, b]\) only if 
\[
\lim_{x \to p^-} h(x) \text{ exists for } a < p < b \text{ and } \lim_{x \to p^+} h(x) \text{ exists for } a < p < b.
\]
Further, \( G \in OL^o \) on \([a, b]\) only if 
\[
\lim_{x \to p^-} G(x, p) \text{ and } \lim_{x, y \to p^-} G(x, y) \text{ exist for } a < p \leq b,
\]
and 
\[
\lim_{x \to p^+} G(p, x) \text{ and } \lim_{x, y \to p^+} G(x, y) \text{ exist for } a \leq p < b.
\]

The theorems in this paper were suggested by a result of J. S. Mac Nerney [5, Theorem 3.4, p. 361]. However, their proofs are constructed by using techniques similar to those employed by B. W. Helton [2, §5, pp. 307–314].

There, the integral equations
\[
fix) = h(x) + \int_a^x f(u)G(u, v) \quad \text{and} \quad fix) = h(x) + \int_a^x G(u, v)f(u)
\]
were solved with the restriction that \( h \) have bounded variation. We now solve these integral equations with the restriction that \( h \) be quasi-continuous. The reader is also referred to related results by D. B. Hinton [4, Theorems 4.1, 4.2, 4.3, 4.4, pp. 325–327] and C. W. Bitzer [1, Theorems 4.1, 4.2, 5.9, pp. 447–451].

We now state three lemmas that are used in the development of the results in this paper.

**Lemma 1.** If \( G \) is a function from \( R \times R \) to \( N \) and \( G \in OB^o \) on \([a, b]\),
then the following statements are equivalent:

1. \( G \in OA^o \) on \([a, b]\), and
2. \( G \in OM^o \) on \([a, b]\) [2, Theorem 3.4, p. 301].

**Lemma 2.** If \( H \) and \( G \) are functions from \( R \times R \) to \( N \), \( H \in OL^o \) on \([a, b]\) and \( G \) is in \( OA^o \) and \( OB^o \) on \([a, b]\), then \( GH \) and \( HG \) are in \( OA^o \) and \( OM^o \) on \([a, b]\) [3, Theorem 2, p. 494].

**Lemma 3.** If \( F \) and \( G \) are functions from \( R \times R \) to \( N \) belonging to \( OB^o \) on \([a, b]\), \( F \in OA^o \) on \([a, b]\) and each of \( \int x \Pi^y (1 + G) \) and \( \int^y_x F(u, v) \Pi^y (1 + G) \) exists for \( a \leq x < y \leq b \), then
\[
\int^b_a \left| F(x, y) - \int^y_x F(u, v) \Pi^y (1 + G) \right|
\]
exists and is zero [2, Lemma, p. 307].
Theorem 1. If \( f \) and \( h \) are functions from \( R \) to \( N \), \( G \) is a function from \( R \times R \) to \( N \), \( h \) is quasi-continuous on \([a, b]\) and \( G \in OB^0 \) on \([a, b]\), then the following statements are equivalent:

1. \( f \) is bounded on \([a, b]\), \( G \in OA^0 \) on \([a, b]\), \( f(u)G(u, v) \in OA^0 \) on \([a, b]\) and \( f(x) = h(x) + \int_a^x f(u)G(u, v) \) for \( a \leq x \leq b \), and

2. \( G \in OM^0 \) on \([a, b]\) and \( \int_a^x h(u)G(u, v) \Pi^x (1 + G) \) exists and is \( f(x) - h(x) \) for \( a \leq x \leq b \).

Proof. (1) \( \rightarrow \) (2). Since \( G \) is in \( OA^0 \) and \( OB^0 \) on \([a, b]\), it follows from Lemma 1 that \( G \in OM^0 \) on \([a, b]\). Suppose \( a \leq x \leq b \). If \( a = x \), then (2) follows immediately. Therefore, suppose \( a < x \). The existence of \( I(a, x) \) follows from Lemma 2, where

\[
I(r, s) = \int_r^s h(u)G(u, v) \Pi^s (1 + G)
\]

for \( a \leq r \leq s \leq b \). We now show that \( I(a, x) \) is equal to \( f(x) - h(x) \). Let \( \epsilon > 0 \).

There exists a subdivision \( D_1 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_1 \), then

\[
\left| I(a, x) - \sum_{i=1}^n h_{i-1}G_i \Pi^x (1 + G) \right| < \epsilon/3.
\]

There exist a subdivision \( D_2 \) of \([a, x]\) and a number \( B \) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_2 \), then

1. \( \Pi_{i=1}^n (1 + |G_i|) < B \), and
2. \( \sum_{i=1}^n |h_{i-1}G_i| < B \).

Since \( G \) is in \( OM^0 \) and \( OB^0 \) on \([a, x]\), there exists a subdivision \( D_3 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_3 \) and \( 1 \leq i \leq n \), then

\[
\left| \frac{1}{x_i} \Pi^{x_i} (1 + G) - \prod_{k=i+1}^n (1 + G_k) \right| < \epsilon(3B)^{-1}.
\]

Since \( f(u)G(u, v) \in OA^0 \) on \([a, x]\), there exists a subdivision \( D_4 \) of \([a', x]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_4 \), then

\[
\sum_{i=1}^n \left| f_{i-1}G_i - \int_{x_{i-1}}^{x_i} f(u)G(u, v) \right| < \epsilon(3B)^{-1}.
\]

Let \( D \) denote the subdivision \( \bigcup_{i=1}^4 D_i \) of \([a, x]\). Suppose \( \{x_i\}_{i=0}^n \) is a refinement of \( D \). For \( 1 \leq i \leq n \), we have that
where
\[ c_i = \int_{x_{i-1}}^x f(u)G(u, v) - f_{i-1}G_i. \]

Thus,
\[ f_i = h_i - h_{i-1} + f_{i-1}(1 + G_i) + c_i. \]

Now, by using iteration for \( i = 1, 2, \ldots, n \), we have that
\[ f_n = h_n + \sum_{i=1}^n h_{i-1}G_i \prod_{k=i+1}^n (1 + G_k) + \sum_{i=1}^n c_i \prod_{k=i+1}^n (1 + G_k). \]

Hence,
\[ |I(a, x) - \{f(x) - h(x)\}| \]
\[ = \left| h(x) + I(a, x) - h_n \right| \]
\[ \leq \left| \sum_{i=1}^n h_{i-1}G_i \prod_{k=i+1}^n (1 + G_k) - \sum_{i=1}^n c_i \prod_{k=i+1}^n (1 + G_k) \right| \]
\[ + \left| I(a, x) - \sum_{i=1}^n h_{i-1}G_i \prod_{i=1}^x (1 + G) \right| + \left| \sum_{i=1}^n [-c_i] \prod_{k=i+1}^n (1 + G_k) \right| \]
\[ < \sum_{i=1}^n |h_{i-1}G_i| \prod_{i=1}^x (1 + G) - \prod_{k=i+1}^n (1 + G_k) + \epsilon/3 + B[\epsilon(3B)^{-1}] \]
\[ < B[\epsilon(3B)^{-1}] + 2\epsilon/3 = \epsilon. \]

Therefore, (1) implies (2).

**Proof.** (2) \( \rightarrow \) (1). Since \( h \) is bounded on \([a, b]\) and \( G \in OB^0\) on \([a, b]\), it follows that \( f \) is bounded on \([a, b]\). Since \( G \) is in \( OM^0\) and \( OB^0\) on \([a, b]\), it follows from Lemma 1 that \( G \in OA^0\) on \([a, b]\). Further, it follows by employing Lemma 2 that \( f(u)G(u, v) \in OA^0\) on \([a, b]\). Suppose \( a \leq x \leq b \). If \( a = x \), then (2) follows immediately. Therefore, suppose \( a < x \). Let \( \epsilon > 0 \).

There exists a subdivision \( D_1 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_1 \), then
\[ \left| \int_a^x f(u)G(u, v) - \sum_{i=1}^n f_{i-1}G_i \right| < \frac{\epsilon}{3}. \]
There exists a number \( B \) such that, if \( a \leq t \leq x \), then \( |I(a, t)| < B \), where \( I(a, t) \) is defined as in the first part of the proof. Now, since \( G \in OM^0 \) on \([a, x]\), there exists a subdivision \( D_2 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^{m} \) is a refinement of \( D_2 \), then
\[
\sum_{i=1}^{n} \left| 1 + G_i - \prod_{i=1}^{x_i} (1 + G) \right| < \epsilon (3B)^{-1}.
\]

It follows by applying Lemma 3 that there exists a subdivision \( D_3 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^{m} \) is a refinement of \( D_3 \), then
\[
\sum_{i=1}^{n} |h_{i-1} G_i - I(x_{i-1}, x_i)| < \frac{\epsilon}{3}.
\]

Let \( D \) denote the subdivision \( \bigcup_{i=1}^{3} D_i \) of \([a, x]\). Suppose \( \{x_i\}_{i=0}^{m} \) is a refinement of \( D \). Thus,
\[
\left| h(x) + \int_a^x f(u) G(u, v) - f(x) \right|
\]
\[
\leq \left| h(x) + \sum_{i=1}^{n} f_{i-1} G_i - h(x) - I(a, x) \right| + \left| \int_a^x f(u) G(u, v) - \sum_{i=1}^{n} f_{i-1} G_i \right|
\]
\[
< \sum_{i=1}^{n} \left| h_{i-1} + I(a, x_{i-1}) \right| G_i - I(a, x) \right| + \frac{\epsilon}{3}
\]
\[
= \sum_{i=1}^{n} \left| h_{i-1} G_i + [I(a, x_{i-1})] [1 + G_i] - I(a, x_{i-1}) \right| - I(a, x) \right| + \frac{\epsilon}{3}
\]
\[
\leq \sum_{i=1}^{n} \left| h_{i-1} G_i + [I(a, x_{i-1})] \left[ \prod_{i=1}^{x_i} (1 + G) \right] - I(a, x_{i-1}) \right| - I(a, x) \right|
\]
\[
+ \sum_{i=1}^{n} \left| I(a, x_{i-1}) \right| 1 + G_i - \prod_{i=1}^{x_i} (1 + G) \right| + \frac{\epsilon}{3}
\]
\[
< \sum_{i=1}^{n} \left| I(a, x_i) - I(a, x_{i-1}) \right| - I(a, x) \right|
\]
\[
+ \sum_{i=1}^{n} \left| h_{i-1} G_i - I(x_{i-1}, x_i) \right| + B[\epsilon (3B)^{-1}] + \frac{\epsilon}{3}
\]
\[
< 0 + \frac{\epsilon}{3} + 2\epsilon = \epsilon.
\]

Therefore, (2) implies (1).
Theorem 2. If \( f \) and \( h \) are functions from \( \mathbb{R} \) to \( \mathbb{N} \), \( G \) is a function from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{N} \), \( G(y, x) = G(x, y) \) for \( a \leq x < y \leq b \), \( h \) is quasi-continuous on \([a, b]\) and \( G \in O\mathbb{B}^0 \) on \([a, b]\), then the following statements are equivalent:

1. \( f \) is bounded on \([a, b]\), \( G \in O\mathbb{A}^0 \) on \([a, b]\), \( G(u, v)f(u) \in O\mathbb{A}^0 \) on \([a, b]\) and \( f(x) = h(x) + \int_a^x G(u, v)f(u) \) for \( a \leq x \leq b \), and

2. \( G \in O\mathbb{M}^0 \) on \([b, a]\) and \( \int_a^x \Pi^\nu (1 + G(u, v))h(u) \) exists and is \( f(x) - h(x) \) for \( a \leq x \leq b \).

The proof of Theorem 2 is similar to the proof of Theorem 1, and therefore, we omit it.

REFERENCES


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