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TOPOLOGICAL SPACES THAT ARE α-FAVORABLE FOR A PLAYER WITH PERFECT INFORMATION

H. E. WHITE, JR.

ABSTRACT. The class of spaces mentioned in the title is closely related to the class of α-favorable spaces introduced by G. Choquet [3]. For convenience, call the spaces mentioned in the title weakly α-favorable. The following statements are true: (1) every dense G_δ subset of a quasi-regular, weakly α-favorable space is weakly α-favorable; (2) the product of any family of weakly α-favorable spaces is weakly α-favorable; (3) any continuous, open image of a weakly α-favorable space is weakly α-favorable; (4) a quasi-regular space with a σ-disjoint pseudobase is weakly α-favorable if and only if it is pseudo-complete in the sense of J. C. Oxtoby; and (5) the product of a weakly α-favorable space and a Baire space is a Baire space.

1. **Introduction.** In recent years, a number of classes of topological spaces have been considered, each of which is a subclass of the class of Baire spaces, and each of which is closed under the formation of products (see [1] for a discussion of these classes). The purpose of this note is to show that the class of spaces mentioned in the title has a number of reasonable properties. The author would like to thank D. J. Lutzer for suggesting (in a letter) the desirability of finding a class of spaces satisfying statements which are essentially (2), (4), (5), (6), (7), (8), and (11) of the theorem, and for supplying the author with a copy of [1], which proved very helpful.

2. **Definitions.** For any function ϕ, let $D(\phi)$, $R(\phi)$ denote the domain and the range of ϕ, respectively. For any collection $\mathcal{K}$ of sets, let $\mathcal{K}^* = \mathcal{K} \sim \{\emptyset\}$.

A topological space $(X, \mathcal{T})$ is called weakly α-favorable (or α-favorable for a player with perfect information) if there is a sequence $\mathcal{S} = (\phi_n)_{n \in N}$ of functions such that

(2.1) $D(\phi_1) = \mathcal{T}^* \supset R(\phi_1)$ and $\phi_1(U) \subset U$ for all U in $D(\phi_1)$,

(2.2) for all n in N,

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$$D(\phi_{n+1}) = \{(U_1, \cdots, U_{n+1}) \in (\mathcal{T}^*)^{n+1}:$$
$$U_{j+1} \subset \phi_j(U_1, \cdots, U_j) \text{ for } j = 1, \cdots, n\},$$

$R(\phi_{n+1}) \subset \mathcal{T}^*$, and $\phi_{n+1}(U_1, \cdots, U_{n+1}) \subset U_{n+1}$ for all $(U_1, \cdots, U_{n+1})$ in $D(\phi_{n+1})$, and

(2.3) if $(U_n)_{n \in N}$ is a sequence such that

(2.3.1) $$(U_1, \cdots, U_n) \in D(\phi_n) \quad \text{for all } n \text{ in } N,$$

then $\bigcap\{U_n : n \in N\} \neq \emptyset$.

There is an interesting discussion in [3, pp. 115–116] which can be used to give an interpretation, in the language of game theory, of the preceding definition.

Any sequence $\mathcal{S} = (\phi_n)_{n \in N}$ which satisfies (2.1), (2.2), and (2.3) is called a winning strategy for $(X, \mathcal{T})$. A sequence $(U_n)_{n \in N}$ which satisfies (2.3.1) is called an $\mathcal{S}$-sequence.

A subfamily $\mathcal{P}$ of $\mathcal{T}$ is called a pseudo-base for $\mathcal{T}$ if every nonempty element of $\mathcal{T}$ contains a nonempty element of $\mathcal{P}$. A pseudo-base $\mathcal{P}$ is called σ-disjoint if $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in N\}$, where each $\mathcal{P}_n$ is a disjoint family.

3. **Theorem.** *Suppose $(X, \mathcal{T})$ is a topological space.*

(1) *If X is weakly α-favorable, then X is a Baire space.*

(2) *If X is locally weakly α-favorable, then X is weakly α-favorable.*

(3) *If X is either pseudo-complete* [4] *or α-favorable* [3], *then X is weakly α-favorable.*

(4) *If, for each i in I, X_i is weakly α-favorable and $\mathfrak{m} \geq \aleph_0$, then the $\mathfrak{m}$-box product of $(X_i)_{i \in I}$ is weakly α-favorable.*

(5) *If X is weakly α-favorable and U is open in X, then U is weakly α-favorable.*

(6) *If X is quasi-regular* [4] *and weakly α-favorable, and X_0 is a dense G_δ subset of X, then X_0 is weakly α-favorable.*

(7) *If θ is a continuous, closed, irreducible mapping of $(X, \mathcal{T})$ onto $(Y, \mathcal{U})$, then X is weakly α-favorable if and only if Y is weakly α-favorable.*

(8) *If θ is a continuous, open mapping of $(X, \mathcal{T})$ onto $(Y, \mathcal{U})$, and X is weakly α-favorable, then Y is weakly α-favorable.*

(9) *If X is a Baire space and $(Y, \mathcal{U})$ is weakly α-favorable, then $X \times Y$ is a Baire space.*

(10) *If X has a σ-disjoint pseudo-base $\mathcal{P}$, then X is weakly α-favorable if and only if it is α-favorable. If X is also quasi-regular, then X is weakly α-favorable if and only if X is pseudo-complete.*

(11) *If X is a T_0 space with a base of countable order* [5], *then X is weakly α-favorable if and only if there is a dense G_δ subset which is metrically topologically complete.*

Remarks. (i) It follows from (10) that the concepts of weakly α-favorable, α-favorable, and pseudo-complete coincide for the class of quasi-regular spaces which have dense metrizable subspaces. In particular, they coincide for quasi-regular, semi-metrizable spaces (since every semi-metrizable Baire space has a dense metrizable subspace).

(ii) It follows from (4) and (8) that $X \times Y$ is weakly α-favorable if and only if both X and Y are weakly α-favorable.

(iii) Statement (9) generalizes 4.2 of [2]. The proof given here is shorter and simpler than the proof of 4.2 that is given in [2].

(iv) Statement (11) is very similar to the corollary to Theorem 2.4 of [2].

(v) A generalization of Theorem 2.4 of [2] can be obtained by combining (3), (8), and (10).

Proof. The proofs of (1), (2), (3), and (5) are easy and are omitted. The proof of (4) is quite similar to the proof of theorem 7.12(iv) of [3], and is omitted.

(6) Suppose $\mathcal{S} = (\phi_n)_{n\in N}$ is a winning strategy for $(X, \mathcal{T})$. Since $\mathcal{T}$ is quasi-regular, the family $\mathcal{R}$ of all regular elements of $\mathcal{T}$ is a pseudo-base for $\mathcal{T}$; hence we may assume that $\bigcup\{R(\phi_n): n \in N\} \subset \mathcal{R}$.

Suppose $X_0 = \bigcap\{G_n: n \in N\}$, where each G_n is open in X. Denote the relative topology on X_0 by $\mathcal{T}_0$, and define $\gamma: \mathcal{T}_0 \to \mathcal{T}$ so that $X_0 \cap \gamma(U) = U$ for all U in $\mathcal{T}_0$. Define, by induction, a sequence $\mathcal{S}_0 = (\psi_n)_{n\in N}$ which satisfies (2.1) and (2.2) relative to $(X_0, \mathcal{T}_0)$, and such that if $n \in N$ and $(U_1, \cdots, U_n) \in D(\psi_n)$, then $(G_1 \cap \gamma(U_1), \cdots, G_n \cap \gamma(U_n)) \in D(\phi_n)$ and

$$(3.1) \qquad \psi_n(U_1, \cdots, U_n) = X_0 \cap \phi_n(G_1 \cap \gamma(U_1), \cdots, G_n \cap \gamma(U_n)).$$

In detail: Suppose $n > 1$, and that, for $j = 1, \cdots, n-1$, ψ_j has been defined. Suppose $(U_1, \cdots, U_n) \in (\mathcal{T}_0^*)^n$ and $U_{j+1} \subset \psi_j(U_1, \cdots, U_j)$ for $j = 1, \cdots, n-1$. Then

$$\begin{aligned} X_0 \cap \gamma(U_n) &\subset \psi_{n-1}(U_1, \cdots, U_{n-1}) \\ &\subset \phi_{n-1}(G_1 \cap \gamma(U_1), \cdots, G_{n-1} \cap \gamma(U_{n-1})). \end{aligned}$$

Since $R(\phi_{n-1}) \subset \mathcal{R}$ and X_0 is dense in X, $\gamma(U_n)$ is contained in $\phi_{n-1}(G_1 \cap \gamma(U_1), \cdots, G_{n-1} \cap \gamma(U_{n-1}))$. Therefore $\psi_n(U_1, \cdots, U_n)$ can be defined by (3.1).

Now, if $(U_n)_{n\in N}$ is an $\mathcal{S}_0$-sequence, then $(G_n \cap \gamma(U_n))_{n\in N}$ is an $\mathcal{S}$-sequence. Hence $\bigcap\{U_n : n \in N\} = \bigcap\{G_n \cap \gamma(U_n): n \in N\} \neq \varnothing$, and $\mathcal{S}_0$ is a winning strategy for $(X_0, \mathcal{T}_0)$.

(7) Define $\theta^*: \mathcal{T}^* \to \mathfrak{U}^*$ by letting $\theta^*(U) = Y \sim \theta[X \sim U]$ for each U in $\mathcal{T}^*$.

If $(X, \mathcal{T})$ is weakly α-favorable and $\mathcal{S} = (\phi_n)_{n\in N}$ is a winning strategy for $(X, \mathcal{T})$, then we define, by induction, a sequence $\mathcal{S}_0 = (\psi_n)_{n\in N}$ which satisfies (2.1) and (2.2) relative to $(Y, \mathfrak{U})$ and such that, if $n \in N$ and $(V_1, \cdots, V_n) \in D(\psi_n)$, then $(\theta^{-1}[V_1], \cdots, \theta^{-1}[V_n]) \in D(\phi_n)$ and

$$\psi_n(V_1, \cdots, V_n) = \theta^*(\phi_n(\theta^{-1}[V_1], \cdots, \theta^{-1}[V_n])).$$

If $(V_n)_{n\in N}$ is an $\mathcal{S}_0$-sequence, then $(\theta^{-1}[V_n])_{n\in N}$ is an $\mathcal{S}$-sequence, so $\bigcap\{V_n : n \in N\} = \theta[\bigcap\{\theta^{-1}[V_n]: n \in N\}] \neq \varnothing$.

If $(Y, \mathfrak{U})$ is weakly α-favorable and $\mathcal{S}_0 = (\psi_n)_{n\in N}$ is a winning strategy for $(Y, \mathfrak{U})$, then we define, by induction, a sequence $\mathcal{S} = (\phi_n)_{n\in N}$ which satisfies (2.1) and (2.2) and such that, if $n \in N$ and $(U_1, \cdots, U_n) \in D(\phi_n)$, then $(\theta^*(U_1), \cdots, \theta^*(U_n)) \in D(\psi_n)$ and

$$\phi_n(U_1, \cdots, U_n) = \theta^{-1}[\psi_n(\theta^*(U_1), \cdots, \theta^*(U_n))].$$

If $(U_n)_{n\in N}$ is an $\mathcal{S}$-sequence, then $(\theta^*(U_n))_{n\in N}$ is an $\mathcal{S}_0$-sequence, so

$$\bigcap\{U_n : n \in N\} \supset \bigcap\{\theta^{-1}[\theta^*(U_n)]: n \in N\} = \theta^{-1}\left[\bigcap\{\theta^*(U_n): n \in N\}\right] \neq \varnothing.$$

(8) Suppose $\mathcal{S} = (\phi_n)_{n\in N}$ is a winning strategy for $(X, \mathcal{T})$. Define, by induction, sequences $\mathcal{S}_0 = (\psi_n)_{n\in N}$, $(\gamma_n)_{n\in N}$ of functions such that (a) $\mathcal{S}_0$ satisfies (2.1) and (2.2) relative to $(Y, \mathfrak{U})$, (b) $D(\gamma_1) = D(\psi_1)$ and, if $V \in D(\gamma_1)$, then $\gamma_1(V) = \theta^{-1}[V]$ and $\psi_1(V) = \theta[\phi_1(\gamma_1(V))]$, and (c) for all n in N, (n.1) $D(\gamma_{n+1}) = D(\psi_{n+1})$ and $R(\gamma_{n+1}) \subset D(\phi_{n+1})$, and (n.2) if $(V_1, \cdots, V_{n+1}) \in D(\gamma_{n+1})$ and $\gamma_n(V_1, \cdots, V_n) = (U_1, \cdots, U_n)$, then

$$\gamma_{n+1}(V_1, \cdots, V_{n+1}) = (U_1, \cdots, U_n, \theta^{-1}[V_{n+1}] \cap \phi_n(U_1, \cdots, U_n))$$

and

$$\psi_{n+1}(V_1, \cdots, V_{n+1}) = \theta[\phi_{n+1}(\gamma_{n+1}(V_1, \cdots, V_{n+1}))].$$

Now, if $(V_n)_{n\in N}$ is an $\mathcal{S}_0$-sequence, then there is an $\mathcal{S}$-sequence $(U_n)_{n\in N}$ such that $\theta[U_n] = V_n$ for all n in N. Hence $\bigcap\{V_n : n \in N\} \supset \theta[\bigcap\{U_n : n \in N\}] \neq \varnothing$.

(9) Suppose $\mathcal{S} = (\phi_n)_{n\in N}$ is a winning strategy for $(Y, \mathfrak{U})$ and that

$(G_n)_{n\in N}$ is a sequence of dense, open subsets of $X \times Y$. It suffices to show that $\bigcap\{G_n : n \in N\} \neq \emptyset$.

We define, by induction, a sequence $(\gamma_n)_{n\in N}$ of functions such that, for each n in N, (n.1) $D(\gamma_n)$ is a disjoint subfamily of $\mathcal{T}^*$, $R(\gamma_n) \subset \mathfrak{U}^*$, and $H \times \gamma_n(H) \subset G_n$ for all H in $D(\gamma_n)$, (n.2) $D(\gamma_{n+1})$ refines $D(\gamma_n)$, (n.3) if $H_j \in D(\gamma_j)$ for $j = 1, \cdots, n$, and $H_{j+1} \subset H_j$ for $j = 1, \cdots, n-1$, then $(\gamma_1(H_1), \cdots, \gamma_n(H_n)) \in D(\phi_n)$, and (n.4) $\bigcup D(\gamma_n)$ is dense in X. In detail: At the nth step, let $\mathcal{F}_n$ denote the set of all functions γ such that (n.1) and (n.3) (and, if $n > 1$, ((n − 1).2)), with γ_n replaced by γ, hold. Order $\mathcal{F}_n$ by inclusion. Since Zorn's lemma is applicable, there is a maximal element γ_n in $\mathcal{F}_n$. If γ_n does not satisfy (n.4) and $n > 1$, then, for $j = 1, \cdots, n-1$, there is H_j in $D(\gamma_j)$ such that $H_{j+1} \subset H_j$ for $j = 1, \cdots, n-2$, and $H_{n-1} \cap [X \sim \mathrm{cl}[\bigcup D(\gamma_n)]] \neq \emptyset$. So there are H in $\mathcal{T}^*$, K in $\mathfrak{U}^*$ such that $H \times K$ is contained in

$$G_n \cap \left\{\left[H_{n-1} \cap \left[X \sim \mathrm{cl}\left[\bigcup D(\gamma_n)\right]\right]\right] \times \phi_{n-1}(\gamma_1(H_1), \cdots, \gamma_{n-1}(H_{n-1}))\right\}.$$

Then $\gamma_n \cup (H, K) \in \mathcal{F}_n$. Therefore γ_n satisfies (n.4).

Since X is a Baire space, $\bigcap\{\bigcup D(\phi_n) : n \in N\} \neq \emptyset$. Suppose $x \in \bigcap\{\bigcup D(\phi_n) : n \in N\}$. Then there is a sequence $(H_n)_{n\in N}$ such that for each n in N, $H_n \in D(\gamma_n)$ and $H_{n+1} \subset H_n$. By $((\mathrm{n}.3))_{n\in N}$, $(\gamma_n(H_n))_{n\in N}$ is an $\mathcal{S}$-sequence; hence $\bigcap\{\gamma_n(H_n) : n \in N\} \neq \emptyset$. And, if $y \in \bigcap\{\gamma_n(H_n) : n \in N\}$, then $(x, y) \in \bigcap\{H_n \times \gamma_n(H_n) : n \in N\} \subset \bigcap\{G_n : n \in N\}$.

(10) Suppose $\mathcal{S} = (\phi_n)_{n\in N}$ is a winning strategy for $(X, \mathcal{T})$ and $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in N\}$ where, for each n in N, $\mathcal{P}_n$ is a disjoint family and $\bigcup \mathcal{P}_n$ is dense in X.

We define, by induction, a sequence $(\gamma_n)_{n\in N}$ of functions such that, for each n in N, (n.1) $D(\gamma_n)$ is a disjoint subfamily of $\mathcal{T}^*$ that refines $\mathcal{P}_n$ and $R(\gamma_n) \subset \mathcal{T}^*$, (n.2) $D(\gamma_{n+1})$ refines $D(\gamma_n)$, (n.3) if $H_j \in D(\gamma_j)$ for $j = 1, \cdots, n$, and $H_{j+1} \subset H_j$ for $j = 1, \cdots, n-1$, then $(\gamma_1(H_1), \cdots, \gamma_n(H_n)) \in D(\phi_n)$ and $\phi_n(\gamma_1(H_1), \cdots, \gamma_n(H_n)) = H_n$, and (n.4) $\bigcup D(\gamma_n)$ is dense in X. In detail: At the nth step, let γ_n be a maximal element of $\mathcal{F}_n$, where we define $\mathcal{F}_n$ verbatim as in (9). If γ_n does not satisfy (n.4) and $n > 1$, then there is P_n in $\mathcal{P}_n$ and, for $j = 1, \cdots, n-1$, there is H_j in $D(\gamma_j)$ such that $H_{j+1} \subset H_j$ for $j = 1, \cdots, n-2$ and

$$K = P_n \cap H_{n-1} \cap \left[X \sim \mathrm{cl}\left[\bigcup D(\gamma_n)\right]\right] \neq \emptyset.$$

Then

$$\gamma_n \cup \{(\phi_n(\gamma_1(H_1), \cdots, \gamma_{n-1}(H_{n-1}), K), K)\} \in \mathcal{F}_n.$$

Therefore γ_n satisfies (n.4).

Let $X_0 = \bigcap\{\bigcup D(\gamma_n): n \in N\}$ and $\mathcal{B}_0 = \{H \cap X_0: H \in \bigcup\{D(\gamma_n): n \in N\}\}$. Then $\mathcal{B}_0$ is a base for a topology $\mathcal{T}_0$ on X_0. Define the pseudo-metric d on X_0 by letting

$$d(x, y) = \inf\{n^{-1}: \text{there is } H_n \text{ in } D(\gamma_n) \text{ such that } x, y \in H_n\}$$

for all x, y in X_0. Then $\mathcal{T}_0$ is the pseudo-metric topology induced by d. In fact, if $H_n \in D(\gamma_n)$ and $x \in H_n \cap X_0$, then $\{y \in X_0: d(x, y) \leq n^{-1}\} = H_n \cap X_0$. And (X_0, d) is complete. For suppose $(x_n)_{n \in N}$ is a Cauchy sequence in (X_0, d). We may assume that, for all n in N, $d(x_n, x_{n+1}) \leq n^{-1}$. Then there is a sequence $(H_n)_{n\in N}$ such that, for all n in N, $x_n \in H_n \in D(\gamma_n)$ and $H_{n+1} \subset H_n$. By $((\text{n}.3))_{n \in N}$, $(\gamma_n(H_n))_{n \in N}$ is an $\mathcal{S}$-sequence; hence $\bigcap\{\gamma_n(H_n): n \in N\} \neq \emptyset$. But, since $\gamma_{n+1}(H_{n+1}) \subset H_n$ for each n, $\bigcap\{H_n: n \in N\} \neq \emptyset$. And, if $x \in \bigcap\{H_n: n \in N\}$, then $(x_n)_{n \in N}$ converges to x. Hence $(X_0, \mathcal{T}_0)$ is both α-favorable and pseudo-complete. Since $\mathcal{T}_0$ is a pseudo-base for the relative topology $\mathcal{T}(X_0)$ on X_0, $(X_0, \mathcal{T}(X_0))$ is both α-favorable and pseudo-complete. Since X_0 is dense in X, $(X, \mathcal{T})$ is α-favorable and, if $\mathcal{T}$ is quasi-regular, $(X, \mathcal{T})$ is pseudo-complete.

(11) The proof of (11) is very similar to the proof of (10). In fact, if $(X, \mathcal{T})$ is a T_0 space without any isolated points, and $\mathcal{B}$ is a base of countable order for $\mathcal{T}$, then we can choose the sequence $(\mathcal{P}_n)_{n \in N}$ so that $\bigcup\{\mathcal{P}_n: n \in N\} \subset \mathcal{B}$ and $\mathcal{P}_n \cap \mathcal{P}_{n+1} = \emptyset$ for all n in N. Then it is easily verified that $\mathcal{T}_0 = \mathcal{T}(X_0)$; hence X_0 is the required metrically topologically complete dense G_δ.

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251 NORTH BLACKBURN ROAD, RT. 5, ATHENS, OHIO 45701