NONCONTINUITY OF TOPOLOGICAL ENTROPY OF MAPS
OF THE CANTOR SET AND OF THE INTERVAL

LOUIS BLOCK

ABSTRACT. We show that topological entropy, as a map on the space of continuous functions of the Cantor set into itself, is not continuous anywhere. Furthermore, topological entropy, as a map on the space of continuous functions of the interval into itself, is not continuous at any map with finite entropy.

1. Introduction. For a compact topological space $S$, let $C^0(S, S)$ denote the space of continuous functions of $S$ into itself with the topology of uniform convergence. Let $I$ denote the unit interval $[0, 1]$ and $C$ the Cantor set (the usual middle third Cantor set). For $f \in C^0(S, S)$, let $\text{ent}(f)$ denote the topological entropy of $f$ as defined in [1]. (We review the definition in §2.) $\text{ent}(f)$ is a nonnegative real number, or $\infty$, which describes (quantitatively) the action of $f$ considered as a discrete dynamical system.

Our main results are the following:

**Theorem A.** The function $\text{ent}: C^0(C, C) \rightarrow \mathbb{R} \cup \{\infty\}$ is not continuous anywhere.

**Theorem B.** The function $\text{ent}: C^0(I, I) \rightarrow \mathbb{R} \cup \{\infty\}$ is not continuous at any map $f$ with $\text{ent}(f)$ finite.

We note that Theorem B is valid with $I$ replaced by the circle $S^1$. (See remarks at the end of §4.)

Topological entropy has been studied in [4], [5] and [7] in connection with Smale's program [8] for studying the orbit structure of differentiable maps of manifolds. However the definition and basic properties rely only on continuity (see [1]). Thus it seems natural to determine what is true in the continuous case before proceeding to the differentiable case.

There are examples on higher dimensional manifolds (see [6]) to show that entropy is not continuous in the differentiable case. However, for the
circle or the interval the problem is open. In this connection we mention the following. Let \( C^1(M, M) \) denote the space of continuously differentiable maps of a compact manifold \( M \) into itself with the \( C^1 \) topology.

**Theorem C.** The function \( \text{ent} : C^1(M, M) \to \mathbb{R} \) is continuous at the identity map of \( M \).

This follows from Proposition 12 of [5].

Finally we remark that positive results on continuity of entropy would have obvious consequences in the theory of bifurcations of differentiable maps. See [2] for some results in this direction.

2. Preliminary definitions and results. We begin by reviewing the definition of topological entropy as defined in [1]. Let \( X \) be a compact topological space. For any two open covers \( \mathcal{V} \) and \( \mathcal{U} \) of \( X \), let \( \mathcal{V} \cap \mathcal{U} \) denote \( \{ A \cap B \mid A \in \mathcal{V} \text{ and } B \in \mathcal{U} \} \). Let \( N(\mathcal{V}) \) denote the number of sets in a subcover of \( \mathcal{V} \) of minimum cardinality.

Let \( f \in C^0(X, X) \). For each integer \( n > 0 \) let

\[
M_n(\mathcal{V}) = N(\mathcal{V} \cup f^{-1}(\mathcal{V}) \cup \cdots \cup f^{-n}(\mathcal{V})).
\]

Here \( f^{-1}(\mathcal{V}) \) denotes the open cover \( \{ f^{-1}(A) \mid A \in \mathcal{V} \} \), and \( f^n \) is defined inductively by \( f^1 = f \) and \( f^n = f^{n-1} \circ f \) for \( n > 1 \).

Set

\[
\text{ent}(f, \mathcal{V}) = \lim_{n \to \infty} n^{-1} \log M_n(\mathcal{V}).
\]

It is easy to show that this limit exists and is finite (see [1]). Finally we define the topological entropy of \( f \) by \( \text{ent}(f) = \sup \{ \text{ent}(f, \mathcal{V}) \mid \mathcal{V} \text{ an open cover of } X \} \).

Next we define the notion of nonwandering set. Let \( f \in C^0(X, X) \). A point \( x \in X \) is said to be wandering if there is a neighborhood \( 0 \) of \( x \) such that \( f^n(0) \cap 0 = \emptyset \) for each integer \( n > 0 \). The set of points which are not wandering is called the nonwandering set and denoted \( \Omega(f) \). We remark that \( \Omega(f) \) is a closed subset of \( X \) and \( f(\Omega(f)) \subset \Omega(f) \).

The following proposition is proved by Bowen in [4]. Here \( X \) is a compact metric space.

**Proposition 1.** Let \( f \in C^0(X, X) \). Then \( \text{ent}(f) = \text{ent}(f|\Omega(f)) \).

One of the inequalities necessary for Proposition 1 follows immediately from the following basic fact which is proved in [1].

**Proposition 2.** Let \( f \in C^0(X, X) \) and let \( K \) be a closed subset of \( X \).
such that $f(K) \subseteq K$. Then $\text{ent}(f) \geq \text{ent}(f|K)$.

It follows immediately from the definition that if $K$ is finite and $f \in C^0(K, K)$ then $\text{ent}(f) = 0$. Hence by Proposition 1 we have

**Proposition 3.** Let $f \in C^0(X, X)$. If $\Omega(f)$ is finite then $\text{ent}(f) = 0$.

From the definition of $\Omega(f)$ it follows that $\Omega(f) \subseteq \text{Im}(f)$ (the image of $f$). Hence we have

**Proposition 4.** Let $f \in C^0(X, X)$. If $\text{Im}(f)$ is finite then $\text{ent}(f) = 0$.

3. **Proof of Theorem A.** We may think of the Cantor set $C$ as the set of infinite sequences $(x_1, x_2, \ldots)$ such that each $x_k$ is 1 or 2. The topology on $C$ is then given by the metric

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} (2^{-i})|x_i - y_i|.$$ (Equivalently we are thinking of $C$ as the infinite product of the set $\{1, 2\}$ with the product topology.)

Let $f \in C^0(C, C)$. We have two cases.

**Case 1.** $\text{ent}(f) > 0$.

Define a sequence $(f_k)$ of functions in $C^0(C, C)$ as follows. Let

$$f_k(x_1, x_2, \ldots) = (y_1, y_2, \ldots, y_k, 1, 1, 1, \ldots)$$

where $f(x_1, x_2, \ldots) = (y_1, y_2, \ldots)$. In other words $f_k$ is the function which assigns to a sequence $(x_1, x_2, \ldots)$ the sequence whose first $k$ terms are the first $k$ terms of $f(x_1, x_2, \ldots)$ and whose terms past the $k$th term are all 1.

Note that the image of the map $f_k$ is a finite set consisting of at most $2^k$ points. (For example the image of $f_2$ consists at most of the points $(1, 1, 1, 1, \ldots), (1, 2, 1, 1, 1, \ldots), (2, 1, 1, 1, 1, \ldots)$, and $(2, 2, 1, 1, 1, \ldots)$.) Hence by Proposition 4, $\text{ent}(f_k) = 0$.

It is easy to see that the sequence $(f_k)$ converges uniformly to $f$. In fact if $\epsilon > 0$, we can choose an integer $N$ large enough to insure that

$$\sum_{k=N+1}^{\infty} 2^{-k} < \epsilon.$$ Then for $k \geq N$,

$$d(f_k(x_1, x_2, \ldots), f(x_1, x_2, \ldots)) < \epsilon$$

for any $(x_1, x_2, \ldots) \in C$.

**Case 2.** $\text{ent}(f) = 0$.

Define a sequence $(g_k)$ of functions in $C^0(C, C)$ as follows. Let

$$f(x_1, x_2, \ldots) = (y_1, y_2, \ldots)$$

and set
In other words, $g_k(x_1, x_2, \cdots)$ is the sequence whose first $k$ terms are the same as the first $k$ terms of $f(x_1, x_2, \cdots)$, and whose $n$th term for $n > k$ is $x_{n+1}$.

As in Case 1, it is clear that $(g_n)$ converges uniformly to $f$. We conclude the proof by showing that for each integer $k > 0$, $\text{ent}(g_k) \geq \log(2)$. Fix $k > 0$.

Let $O_1$ be the set of sequences $(x_1, x_2, \cdots)$ such that $x_{k+1} = 1$. Let $O_2$ be the set of sequences $(x_1, x_2, \cdots)$ such that $x_{k+1} = 2$. Then $\mathcal{G} = \{O_1, O_2\}$ is an open cover of $C$. We will show that $\text{ent}(g_k, \mathcal{G}) = \log(2)$.

Let $x = (x_1, x_2, \cdots) \in C$. Then

$x \in O_1 \cap g_k^{-1}(O_1) \iff x_{k+1} = 1$ and $x_{k+2} = 1,$

$x \in O_1 \cap g_k^{-1}(O_2) \iff x_{k+1} = 1$ and $x_{k+2} = 2,$

$x \in O_2 \cap g_k^{-1}(O_1) \iff x_{k+1} = 2$ and $x_{k+2} = 1,$

$x \in O_2 \cap g_k^{-1}(O_2) \iff x_{k+1} = 2$ and $x_{k+2} = 2.$

Thus the sets $O_1 \cap g_k^{-1}(O_1), O_1 \cap g_k^{-1}(O_2), O_2 \cap g_k^{-1}(O_1),$ and $O_2 \cap g_k^{-1}(O_2)$ are pairwise disjoint nonempty subsets of $C$. Hence $M(g_k, \mathcal{G}) = 4$. It follows in the same way by induction that $M_n(g_k, \mathcal{G}) = 2^n + 1$ for each integer $n > 0$. Hence $\text{ent}(g_k, \mathcal{G}) = \log(2)$. This implies that $\text{ent}(g_k) \geq \log(2)$, and completes the proof of Theorem A.

We remark that since the diameter of $Q \cup g_k \cup g_k^2 \cup \cdots$ approaches zero (as $k \to \infty$), it actually follows that $\text{ent}(g_k) = \log(2)$.

4. Proof of Theorem B. Let $K$ denote any closed interval on the real line. We may form the middle third Cantor subset of $K$, which we denote by $C$, and we may identify points in $C$ with sequences whose terms are all 1 or 2, as in §3.

Let $s$ denote the map in $C^0(C, C)$ defined by $s(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots)$. $s$ is sometimes called the full 2-shift (see [8] for discussion and further references). We will use the following elementary facts (see [1]).

Proposition 5. $\text{ent}(s) = \log(2)$.

Proposition 6. If $f \in C^0(X, X)$ for any compact space $X$, then $\text{ent}(f^n) = n \cdot \text{ent}(f)$.  

We will use the usual metric $d$ on $C^0(I, I)$ which may be defined by

$$d(f, g) = \sup \{|f(x) - g(x)| : x \in I\}.$$  

**Theorem B.** The function $\ent: C^0(I, I) \to R \cup \{\infty\}$ is not continuous at any map $f$ with $\ent(f)$ finite.

**Proof.** Let $f \in C^0(I, I)$ with $\ent(f)$ finite. Let $\ent(f) = \log(K)$. Pick an integer $m > 0$, such that $2^m > 2K$.

Let $x_0$ be a fixed point of $f$. We assume for simplicity that $x_0 \neq 1$. (The proof can be easily modified for the case $x_0 = 1$.)

Let $\epsilon > 0$. There exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - x_0| < \epsilon/2$. We may choose $\delta$ so that $\delta < \epsilon/2$ and $x_0 + \delta < 1$.

We construct a map $g \in C^0(I, I)$ such that $d(f, g) < \epsilon$. We first construct $g$ on the interval $[x_0, x_0 + \delta/2]$ as follows. Let $C$ denote the middle third Cantor subset of the interval $[x_0, x_0 + \delta/2]$. Define $g$ on $C$ by $g = s^m$, where $s$ denotes the full 2-shift as defined above. Note that $g(x_0) = x_0$ and $g(x_0 + \delta/2) = x_0 + \delta/2$ since $x_0$ is identified with the sequence $(1, 1, 1, \ldots)$ and $x_0 + \delta/2$ is identified with the sequence $(2, 2, 2, \ldots)$. We extend $g$ to the interval $[x_0, x_0 + \delta/2]$ by defining $g$ linearly on each open interval in $[x_0, x_0 + \delta/2] - C$.

Next we extend $g$ to the interval $[x_0, x_0 + \delta]$ by defining $g$ on the interval $[x_0 + \delta/2, x_0 + \delta]$ as follows. Let $g(x_0 + \delta/2) = x_0 + \delta/2$, $g(x_0 + \delta) = f(x_0 + \delta)$, and define $g$ linearly on $[x_0 + \delta/2, x_0 + \delta]$. Finally we extend $g$ to a map in $C^0(I, I)$ by defining $g(x) = f(x)$ for $x \in I - [x_0, x_0 + \delta]$.

Note that

$$\ent(g) \geq \ent(s^m) = \log(2^m) > \log(2K) = \log(2) + \log(K).$$

We must show that for all $x \in I$, $|f(x) - g(x)| < \epsilon$.

If $x \in I - [x_0, x_0 + \delta]$ then $|f(x) - g(x)| = 0$. If $x \in [x_0, x_0 + \delta]$ then

$$|g(x) - f(x)| = |g(x) - x_0| + |f(x) - x_0| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Here we have used the fact that $g$ is defined linearly on $[x_0 + \delta/2, x_0 + \delta]$, and $|g(x_0 + \delta/2) - x_0| < \epsilon/2$, and $|g(x_0 + \delta) - x_0| < \epsilon/2$.

We have constructed a map $g \in C^0(I, I)$ such that $d(f, g) < \epsilon$, and $\ent(g) > \ent(f) + \log(2)$. Since $\epsilon$ was arbitrary this completes the proof that $\ent$ is not continuous at $f$. Q.E.D.

We conclude this section by remarking that Theorem B is valid with $I$ replaced by the circle $S^1$. We use the fact that a dense set of maps in
$C^0(S^1, S^1)$ have periodic points (see [3]).

Let $f \in C^0(S^1, S^1)$ and $\epsilon > 0$. Let $f_1 \in C^0(S^1, S^1)$ such that $f_1$ has a periodic point and $d(f, f_1) < \epsilon/2$. By modifying the argument of Theorem B, with a periodic orbit replacing the role of the fixed point, we construct a map $g$ with $\text{ent}(g) > \text{ent}(f) + \log(2)$, and $d(f_1, g) < \epsilon/2$. Hence ent is not continuous at $f$.

5. An example. We close by giving an example of $f \in C^0(I, I)$ such that $\text{ent}(f)$ is infinite.

Let $K_n$ denote the interval $[1/(n+1), 1/n]$ for each integer $n > 0$. Define $f$ on each interval $K_n$ as follows. Let $C_n$ denote the middle third Cantor subset of $K_n$. Let $f = s^n$ on $C_n$ (again $s$ denotes the full 2-shift defined in §4) and extend $f$ to $K_n$ by defining $f$ linearly on each open interval in $K_n - C_n$. We extend $f$ to a map in $C^0(I, I)$ by setting $f(0) = 0$.

It follows from Propositions 2, 5, and 6 that $\text{ent}(f) = \infty$.

REFERENCES


