LOCALIZATIONS AND EVALUATION SUBGROUPS

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ABSTRACT. If $G_n(X)$ is the $n$th evaluation subgroup of a simple connected finite CW-complex, then $G_n(X_p) \cong G_n(X)$ for $p = 0$ or a prime.

Let $X$ be a connected simple finite CW-complex, $X_p$ its localization at $p$ for $p$ prime or $0$ [5], and $G_n(X)$ the evaluation subgroup of $\pi_n(X)$ [1]. If $e_p : X \rightarrow X_p$ is the canonical map, we show that $\alpha \in G_n(X)$ if and only if $e_p^*(\alpha) \in G_n(X_p)$ for all $p$. As corollaries we obtain that $G_n(X_p) \cong G_n(X_p)$ and $X$ is a $G$-space [1] if and only if $X_p$ is a $G$-space for all $p$, where $G_n(X_p)$ is the localization of the group $G_n(X)$. This is analogous to results obtained for $H$-spaces [3], [5]. All $H$-spaces are $G$-spaces, and many properties of $H$-spaces are shared by $G$-spaces.

1. Preliminaries. Spaces $X$ and $W$ are assumed to be pointed, simple (abelian fundamental groups acting trivially on the homotopy and homology groups), connected, finite CW-complexes. We will not distinguish between a map and its homotopy class. For $p$ a prime let $Q_p = \{ k/q | k, q \text{ integers, } p \nmid q \}$ and $Q_0$ the rationals. $Q_p$ is the localization of the integers at the prime $p$. The general reference for localization theory is [5]. We review some of these results here.

Definition 1.1. A space $X$ is $p$-local if $\pi_*(X)$ admits a $Q_p$-module structure extending the usual $\mathbb{Z}$-module structure.

For each $X$ there is a $p$-local space $X_p$ and canonical map $e_p : X \rightarrow X_p$ such that if $g : X \rightarrow Y$, where $Y$ is $p$-local, there is a unique (up to homotopy) $g' : X_p \rightarrow Y$ such that $g \cong g' e_p$. This is equivalent to the map $\phi_p : \pi_*(X) \otimes Q_p \rightarrow \pi_*(X_p)$ being an isomorphism, where $\phi_p(\alpha \otimes r) = r e_p^*(\alpha)$, where the multiplication is the $Q_p$-module structure on $\pi_*(X)$ ($e_p$ $p$-localizes in homotopy). Finally we point out that localization is functorial.

Key results about the evaluation subgroups can be found in [1]. We will establish the notations needed in this paper. $L(W, X; f)$ will be the
path component containing \( f : W \to X \) in the space of functions from \( W \) to \( X \) with the compact-open topology; \( L_0(W, X; f) \) will be the subspace of \( L(W, X; f) \) containing base point preserving functions; \( \omega : L(W, X; f) \to X \) will be the function which evaluates each map at the base point of \( W \).

**Definition 1.2.** The evaluation subgroup, \( G_n(X) \), of \( \pi_n(X) \) is defined by

\[
G_n(X) = \text{Im}(\omega_* : \pi_n(L(X, X; 1_X)) \to \pi_n(X))
\]

where \( 1_X \) is the identity on \( X \).

\( G_n(X) \) consists of all elements \( \alpha \in \pi_n(X) \) such that there is a function \( F : X \times S^n \to X \) with \( F|X \cap S^n = 1_X \cap \alpha \). Such a function will be called an affiliated map for \( \alpha \).

2. **Localizations and evaluation subgroups.** Let

\[(1) \quad \hat{e}_p : L(W, X; f) \to L(W, X_p; e_p f) \]

be defined by \( \hat{e}_p(g) = e_p g \).

**Theorem 2.1.** If \( W \) is a connected finite CW-complex and \( X \) a connected simple CW-complex then (1) \( p \)-localizes in homotopy.

**Proof.** By Corollary 1.3 in [3] \( \bar{e}_p = \hat{e}_p|L_0(W, X; f) \) \( p \)-localizes. Consider the following commutative diagram:

\[
\begin{array}{ccc}
L_0(W, X; f) & \xrightarrow{\bar{e}_p} & L_0(W, X_p; e_p f) \\
\downarrow & & \downarrow \\
L(W, X; f) & \xrightarrow{\hat{e}_p} & L(W, X_p; e_p f) \\
\downarrow & & \downarrow \\
X & \xrightarrow{e_p} & X_p \\
\end{array}
\]

Since \( \omega \) is a fibration in both columns and two of the three horizontal maps \( p \)-localize, \( \hat{e}_p \) also \( p \)-localizes (see 2.21 and [5]).

**Theorem 2.2.** If \( \alpha \in G_n(X) \) then \( e_p*(\alpha) \in G_n(X_p) \).

**Proof.** Let \( F : X \times S^n \to X \) be an affiliated map for \( \alpha \). Let \( \overline{F} \) be the composition

\[
\begin{array}{ccc}
X_p \times S^n & \xrightarrow{1 \times e_p} & X_p \times S^n \\
\downarrow & \xrightarrow{F} & \downarrow \\
X_p & \xrightarrow{e_p} & X_p \\
\end{array}
\]

Note that \( (X \times S^n)_p = X_p \times S^n_p \). Since \( \overline{F}|S^n = e_p \alpha \circ e_p = e_p \alpha \), \( \overline{F} \) is an affiliated map for \( e_p*(\alpha) \). This result does not require \( X \) to be finite.

**Theorem 2.3.** If \( e_p*(\alpha) \in G_n(X_p) \) for all \( p \) then \( \alpha \in G_n(X) \).
Proof. Case 1. $\alpha$ of finite order. Let $p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}$ be the prime factorization of the order $\alpha$. Let $P_1 = \prod_{i \neq 1} p_i^{m_i}$. Since $p_1^{m_1}$ and $P_1$ are relatively prime, there are integers $r$ and $s$ such that $1 = rp_1^{m_1} + sP_1$; so $\alpha = rp_1^{m_1} \alpha + sP_1 \alpha$. $sP_1 \alpha$ is of order $p_1^{m_1}$ while $rp_1^{m_1} \alpha$ is of order $P_1$. $rp_1^{m_1} \alpha$ can then be written as a sum of its (hence $\alpha$'s) multiples such that one summand is of order $p_2^{m_2}$ and the other of order $\prod_{i > 1} p_i^{m_i}$.

By induction $\alpha = k_1 \alpha + k_2 \alpha + \cdots + k_n \alpha$ where $k_i \alpha$ is of order $p_i^{m_i}$. Since $e_\alpha^p(k_i \alpha) \in G_n(X_p)$ for all $p$, $e_\alpha^p(k_i \alpha) \in G_n(X_p)$. So if we can show the result for $\alpha$ of order $p^m$, $p$ any prime, then each $k_i \alpha$ will be in $G_n(X)$ and thus $\alpha \in G_n(X)$.

We assume then that $\alpha$ has order $p^m$ for $p$ prime and localize at this $p$. Consider the following diagram:

$$
\pi_n(L(X, \chi; e_p)) \xrightarrow{\omega} \pi_n(X_p) \xrightarrow{e_p^*} \pi_n(L(X_p^n, X_p^n; 1))
$$

where $e_p^*(f) = fe_p$. From this commutative diagram and the definition of $e_\alpha^p(\alpha) \in G_n(X_p^n)$ it follows that there is an $\tilde{\alpha} \in \pi_n(L(X, \chi; e_p))$ such that $\omega^*(\tilde{\alpha}) = e_\alpha^p(\alpha)$. Let

$$
\tilde{\phi}_p : \pi_n(L(X, \chi; 1)) \otimes Q_p \rightarrow \pi_n(L(X, \chi; e_p))
$$

be the isomorphism guaranteed by Theorem 2.1. There is an element

$$
\sum (\alpha_i \otimes (k_i/q_i)) \in \pi_n(L(X, \chi; 1)) \otimes Q_p
$$

such that $\tilde{\phi}_p(\sum (\alpha_i \otimes (k_i/q_i))) = \tilde{\alpha}$. If $q = \prod q_i$ and $\overline{q_i} = \prod_{i \neq j} q_j$ then

$$
\sum (\alpha_i \otimes (k_i/q_i)) = \sum (\alpha_i \otimes \overline{q_i} k_i/q) = \sum \overline{q_i} k_i \alpha_i \otimes (1/q) = (\sum \overline{q_i} k_i \alpha_i) \otimes (1/q).
$$

Note that $p \nmid q_i$ for each $i$, hence $p \nmid q$. Thus there is an element

$$
\beta \otimes (1/q) \in \pi_n(L(X, \chi; 1)) \otimes Q_p
$$

such that $\tilde{\phi}_p(\beta \otimes (1/q)) = \tilde{\alpha}$ and $p \nmid q$. Consider the following diagram:
By commutativity

\[ \omega_*(\beta \otimes 1) = \omega_*(\phi_p(\beta \otimes 1)) = \omega_*(\phi_p(q\beta \otimes (1/q))) = \omega_*(q\alpha) = e_{p*}(q\alpha). \]

But \( e_{p*}(q\alpha) = \phi_p(q\alpha \otimes 1) \), so \( \omega_*(\beta \otimes 1) = q\alpha \otimes 1 \) since \( \phi_p \) is an isomorphism. But then \( \omega_*(\beta) = q\alpha + \gamma \) where \( \gamma \) has order \( q' \) and \( p \nmid q' \), so \( \omega_*(q\beta) = q'q\alpha \). Since \( p^m \) and \( q'q \) are relatively prime there are integers \( r \) and \( s \) such that \( rp^m + sq'q = 1 \). Thus \( \omega_*(sq'\beta) = sq'q\alpha = \alpha - rp^m\alpha = \alpha \) and \( \alpha \in G_n(X) \).

\textit{Case 2.} \( \alpha \) of infinite order. Localizing at 0 the above argument yields an element

\[ \beta \otimes (1/q) \in \pi_n(L(X, X; 1_X)) \otimes Q_0 \]

such that \( \omega_*(\beta \otimes (1/q)) = \alpha \otimes 1 \), and as above \( \omega_*(q\beta) = q'q\alpha \) for some nonzero integer \( q' \). Thus there are nonzero multiples of \( \alpha \) in \( G_n(X) \). Let \( \overline{q} \) be the least positive integer such that \( \overline{q}\alpha \in G_n(X) \). If \( \overline{q} \neq 1 \) let \( p \) be a prime factor of \( \overline{q} \). Localizing at this \( p \) we obtain an element \( \beta' \in \pi_n(L(X, X; 1_X)) \) and a \( q'' \) where \( p \nmid q'' \) such that \( \omega_*(\beta') = q'' \alpha \). But then \( q''\alpha \) is in the subgroup generated by \( \overline{q}\alpha \) so \( q'' \) is a multiple of \( \overline{q} \). This is a contradiction since \( p \) is a factor of \( \overline{q} \) but not of \( q'' \). Thus \( \overline{q} = 1 \) and \( \alpha \in G_n(X) \).

During the proof we also obtained the following useful corollary.

\textbf{Corollary 2.4.} If \( e_{p*}(\alpha) \in G_n(X_p) \) then there is a \( q \) such that \( p \nmid q \) and \( q\alpha \in G_n(X) \).

\textbf{Corollary 2.5.} \( G_n(X_p) \cong G_n(X)_p \).

\textbf{Proof.} We need only show that \( \phi_p : \pi_n(X) \otimes Q_p \to \pi_n(X_p) \) satisfies

\[ \text{Im} \phi_p|G_n(X) \otimes Q_p = G_n(X_p). \]

Say \( \alpha \otimes (1/q) \in G_n(X) \otimes Q_p \) (we can assume any element in \( G_n(X) \otimes Q_p \) has this form by the technique used in 2.3). Let \( F : X \times S^n \to X \) be an
affiliated map for $s$ and $g: S^n \to S^n$ a map corresponding to $1/q \in \pi_n(S^n) \approx \mathbb{Q}_p$. Then the composition

$$X_p \times S^n \xrightarrow{1 \times g} X_p \times S^n \xrightarrow{F_p} X_p$$

is an affiliated map for $\phi_p(\alpha \otimes (1/q))$ and

$$\text{Im } \phi_p|G_n(X) \otimes \mathbb{Q}_p \subset G_n(X_p).$$

Now say $a \in G_n(X_p)$ and $a \otimes (1/q) = \phi_p^{-1}(\tilde{a})$. Then $e_p*(\alpha) \otimes 1 = q\tilde{a} \otimes 1$ and $e_p*(\alpha) - q\tilde{a}$ is of order $q'$ where $p \nmid q'$. Thus $e_p*(q'\alpha) = q'q\tilde{a} \in G_n(X_p)$. By Corollary 2.5 there is a $a''$ such that $p \nmid q''$ and $q''q\alpha \in G_n(X)$. Then

$$q''q\alpha \otimes \frac{1}{q''q} \in G_n(X) \otimes \mathbb{Q}_p \quad \text{and} \quad \phi_p\left(a''q\alpha \otimes \frac{1}{q''q}\right) = \phi_p\left(a \otimes \frac{1}{q}\right) = a,$$

so $\tilde{a} \in \text{Im } \phi_p|G_n(X) \otimes \mathbb{Q}_p$. Thus $\phi_p|G_n(X) \otimes \mathbb{Q}_p$ is onto $G_n(X_p)$.

**Corollary 2.6.** $X$ is a $G$-space if and only if $X_p$ is a $G$-space for all $p$.

**Proof.** By definition $X$ is a $G$-space if and only if $G_n(X) = \pi_n(X)$ for all $n$. If $X$ is a $G$-space, $X_p$ is a $G$-space for all $p$ by Corollary 2.5. If $X_p$ is a $G$-space for all $p$ then $X$ is a $G$-space by Theorem 2.3.

The following application illustrates how these results can be used.

**Corollary 2.7.** If $G_n(X)$ is torsion without $p$-torsion, any fibration $f: E \to S^{n+1}$ with fiber $X_p$ admits a cross section.

**Proof.** In this case $G_n(X)_p = 0$ so by Corollary 2.5 $G_n(X_p) = 0$ and the result follows from Corollary 2.7 in [1].

For example, if $V$ is the real Stiefel manifold $V_{n-k}$ where $n-k = 1$ or $3$, $k > 1$, any fibration over $S^{n-k+1}$ with fiber $V_{p'}$ $p$ an odd prime, admits a cross section. This follows from Corollary 2.7 and results in [4].

Corollary 2.4 also provides a shortened proof of Proposition 3 in [2] with simply connected replaced by simple. Let $\mathcal{C}$ be a Serre class of Abelian groups. Haslam defines a $G$-space mod $\mathcal{C}$ to be a space $X$ such that $\pi_n(X)/G_n(X) \in \mathcal{C}$. A space will be called an $H$-space mod $\mathcal{C}$ if there is a map $\mu: X \times X \to X$ which, when restricted to each factor, induces a $\mathcal{C}$ isomorphism in homotopy. Let $\mathcal{C}_p$ be the class of finite Abelian groups having no element with order a positive power of $p$; let $\mathcal{C}_0 = \mathcal{F}$ be the class of finite Abelian groups.
Theorem 2.8 (Haslam). If $X$ is an $H$-space mod $C_p$, $p$ prime or 0, then $X$ is a $G$-space mod $C_p$.

Proof. If $X$ is an $H$-space mod $C_p$ then it can be shown that $X_p$ is an $H$-space and hence a $G$-space. For every $\alpha \in \pi_n(X)$, $e_p^*(\alpha) \in G_n(X_p)$ and by Corollary 2.4 there is a $q$ where $p \nmid q$ such that $q\alpha \in G_n(X)$. Thus $X$ is a $G$-space mod $C_p$.

For $C_0$ Haslam has obtained a converse to this theorem [2]. For $C_p$, $p$ prime, the converse is false in general but the simply connected case is still open.

REFERENCES


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