HOMOTOPY SMOOTHING CERTAIN PL-MANIFOLDS
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ABSTRACT. It is shown any simply connected almost parallelizable
PL-manifold of dimension $4k + 2 \neq 2^j - 2 \geq 6$ has the homotopy type of
a smooth manifold if and only if $M$ is stably parallelizable.

Introduction. It has long been known that except in dimensions congruent
to $2 \pmod{4}$ simply connected finite dimensional $h$-spaces have the homotopy
type of smooth stably parallelizable manifolds. (In addition it can be demonstrat-
ated that any stably parallelizable smooth $h$-space manifold is actually
parallelizable.) This paper arises from an attempt to analyze the dimensions
congruent to $2 \pmod{4}$. Although no complete answer to the original problem
is obtained, we believe that the results obtained here will be useful in
further analyzing the problem.

According to Browder and Hirsch [3] any simply connected polyhedron
satisfying Poincaré duality for dim $4k + 2 \geq 5$ whose Spivak normal fibration
is trivial has the homotopy type of a closed PL-manifold $M$ whose normal
bundle restricted to $M - pt$ is trivial (i.e. is almost parallelizable). If $X$ is
a finite dimensional $h$-space, the Spivak normal fibration is trivial [4]. There-
fore one is led to the study of PL-manifolds which are almost parallelizable
and whose stable normal bundle is fiber homotopically trivial. (The second
condition on the normal bundle actually is a consequence of the first.) The
main result of this paper is that if, in addition, $M$ is simply connected of
dimension $4k + 2 \neq 2^j - 2$, $k \geq 1$, then $M$ has the homotopy type of a smooth
manifold if and only if the stable normal bundle of $M$ is trivial. That is, we
obtain an interpretation of the Kervaire invariant as a PL-bundle, which
hopefully will lend itself to geometric analysis, at least in the case of $h$-
spaces.

I. Let $M^n$ be a PL-manifold. From surgery theory one obtains the exact
sequence

$$\mathcal{F}(M) \xrightarrow{\eta} [M, G/PL] \xrightarrow{c} L_n(\pi_1(M)),$$

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where $\mathcal{S}(M)$ is the set of homotopy triangulations of $M$, and $\eta(N, \phi) = (\nu_M - g^*\nu_N, h)$ where $g$ is a homotopy inverse to $\phi$ and $h$ is a stable trivialization of $\nu_M - g^*\nu_N$ determined by $\phi$. For details see [11].

**Lemma 1.** If $M^n$ is a PL-manifold, then $M$ has the homotopy type of a smooth manifold if and only if there exists a vector bundle $\xi$ over $M$ and a lift of $\nu_M - \xi$ to a $G/PL$ bundle $\alpha$ with $c(\alpha) = 0$.

**Proof.** Since $c(\alpha) = 0$, there exists a PL-manifold $N$ and a homotopy equivalence $\phi : N \to M$ with $\nu_N \sim \phi^*(\nu_M - (\nu_M - \xi)) = \phi^*\xi$. Hence $\nu_N$ is a vector bundle and by the smoothing result of Milnor [7], [8], $N$ has a smooth structure compatible with its PL-structure. Conversely if $\phi : N \to M$ is a homotopy equivalence with $N$ smooth, then $\eta(N, \phi) \in [M, G/PL]$ and $\xi = g^*\nu_N$ satisfies the conditions where $g$ is a homotopy inverse to $\phi$. □

**Theorem 1.** Suppose $M^n$ is a simply connected PL-manifold $n = 4k + 2 \neq 2^j - 2$, $k \geq 1$. Further suppose $\nu_M$ is stably fiber homotopically trivial and $\nu_M^{\sim}$ is any lift of $\nu_M$ to a $G/PL$-bundle. Then $M^n$ has the homotopy type of a smooth manifold if and only if $c(\nu_M^{\sim}) = 0$.

**Proof.** By Lemma 1, $M$ has the homotopy type of a smooth manifold if and only if there exists a vector bundle $\xi$ over $M$ and a lift of $\nu_M - \xi$ to a $G/PL$-bundle $\alpha$ with $c(\alpha) = 0$. Consider the $G/PL$-bundle $\nu_M^{\sim} - \alpha$. This projects to a vector bundle and so is a $G/O$-bundle. Sullivan and Rourke have shown [10] there exist classes $\kappa_{4j+2} \in H^{4j+2}(G/PL; \mathbb{Z}_2)$ so that if $g : M \to G/PL$ is any $G/PL$-bundle, $c(g) = \langle V^2(M)g^*(\kappa), [M] \rangle$ where $\kappa = \Sigma\kappa_{4j+2}$ and $V(M)$ is the total Wu-class of $M$. Since $\nu_M$ is fiber homotopically trivial, $V(M) = 1$ and therefore $c(\nu_M^{\sim} - \alpha) = \langle (\nu_M^{\sim} - \alpha)^*\kappa_{4k+2}, [M] \rangle$. Now Madsen, Milgram and Brumfiel [6] have demonstrated that if $4k + 2 \neq 2^j - 2$, $\kappa_{4k+2}$ maps to zero in $H^*(G/O, \mathbb{Z}_2)$. Therefore $c(\nu_M^{\sim} - \alpha) = 0$. But in this dimension $c$ is a homomorphism [11] and so $0 = c(\nu_M^{\sim} - \alpha) = c(\nu_M^{\sim}) - c(\alpha) = c(\nu_M^{\sim})$. For the converse choose $\xi = \text{trivial bundle}$. □

**Proposition.** Suppose $M^n$ is an almost parallelizable PL-manifold, then $\nu_M$ is stably fiber homotopically trivial.

**Proof.** Consider the cofibration $M - pt \hookrightarrow M \xrightarrow{d} S^n$. $\nu_M^{\sim}|M - pt$ is trivial, so there exists a PL-bundle $\eta^q$ over $S^n$ with $d^*\eta = \nu_M$. Since $\nu_M$ is the Spivak normal fibration of $M$, there exists a degree one map $u : S^{n+q} \to T\nu_M$. Further $d$ is a degree one map, and so $Td \circ u : S^{n+q} \to T\eta^q$ is a
degree one map. But this property characterizes the Spivak normal fibration, so \( \eta^g \) is fiber homotopically equivalent to the stable normal bundle of \( S^n \) which is trivial. Therefore \( \nu_M = d^*\eta^g \) is fiber homotopically trivial. □

We are now in a position to prove the main result.

**Theorem 2.** Let \( M^n \) be a closed, simply connected, almost parallelizable PL-manifold with \( n = 4k + 2 \neq 2^j - 2, k \geq 1 \). Then \( M^n \) has the homotopy type of a smooth manifold if and only if \( M^n \) is stably parallelizable.

**Proof.** Since \( M^n \) is almost parallelizable, \( \nu_M \) is stably fiber homotopically trivial. From the Atiyah-Milnor-Spanier theorem \([1],[9]\) it is easy to show there exists a map \( \phi: \Sigma^k S^n \to \Sigma^k M_+ (M_+ = M \text{ with a disjoint base point}) \) such that \( \phi_*(\Sigma^k [S^n]) = \Sigma^k [M^n] \). Consider the cofibration \( M^n - pt \to M^n \to S^n \). Since \( d \) is a degree one map, it follows from the Hopf theorem that \( \Sigma^k d \circ \phi \cong 1d \). Therefore if \( Y \) is any \( k \)-fold loop space \( d^* : [S^n, Y] \to [M^n, Y] \) is a monomorphism onto a direct summand. Since \( \nu_M | M - pt = 0 \) there exists \( \eta \in [S^n, BPL] \) with \( d^* \eta = \nu_M \). Since \( BPL \) is an infinite loop space \([2]\), \( \nu_M = 0 \) if and only if \( \eta = 0 \). As in the proof of the previous proposition, \( \eta \) is stably fiber homotopically trivial and so there exists \( \tilde{\eta} \in [S^n, G/PL] \) lifting \( \eta \). Browder \([5]\) has shown that in the dimensions \( 4k + 2 \neq 2^j - 2 \); the map \([S^{4k+2}, G/PL] \to [S^{4k+2}, BPL] \) is a monomorphism and therefore \( \eta = 0 \) if and only if \( \tilde{\eta} = 0 \). Now \( \pi_{4k+2}(G/PL) \cong \mathbb{Z}_2 \) generated by a map with Arf invariant one, so \( \tilde{\eta} = 0 \) if and only if \( c(\tilde{\eta}) = 0 \). \( \tilde{\eta}d \) is a lift of \( \nu_M \) and since \( d \) has degree one the formula of Rourke and Sullivan for computing \( c \) shows \( c(\tilde{\eta}) = 0 \) if and only if \( c(\tilde{\eta}d) = 0 \). The result now follows from Theorem 1. □

**REFERENCES**

THE INTERSECTION MULTIPLICITY OF 
\textit{n}-DIMENSIONAL PARACOMPACT SPACES

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\textbf{ABSTRACT.} It is shown that there is an integer \( v(n) \leq 3^{2n+1} - 1 \) such that any \( n \)-dimensional paracompact space \( X \) has intersection multiplicity at most \( v(n) \). That is, if \( U \) is an open cover of \( X \), then there is an open cover \( \mathcal{V} \) refining \( U \) such that any element of \( \mathcal{V} \) intersects at most \( v(n) \) elements of \( \mathcal{V} \).

For any open cover \( U \) of a topological space \( X \) define \( m(U) \) to be the maximum number of elements of \( U \) that any member of \( U \) can intersect. \( X \) is said to have intersection multiplicity at most \( m \) if every open cover \( U \) of \( X \) has an open refinement \( \mathcal{V} \) covering \( X \) such that \( m(\mathcal{V}) \leq m \). The intersection multiplicity of \( X \) is then the least integer \( m \) such that \( X \) has intersection multiplicity at most \( m \) and is denoted \( m(X) \). Intersection multiplicity is clearly preserved by homeomorphisms. Also if \( A \) is a closed subset of \( X \), then \( m(A) \leq m(X) \).

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