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SEMILOCAL DOMAINS WHOSE FINITELY GENERATED MODULES ARE DIRECT SUMS OF CYCLICS

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ABSTRACT. A necessary and sufficient condition is given for a semilocal domain to have the property that every finitely generated module is a direct sum of cyclic modules.

Recent work by Tom Shores and Roger Wiegand [SW] and [SWB] has shed new light on the old question: "What is the class of commutative rings for which every finitely generated module is a direct sum of cyclic submodules?" Following their terminology, such a ring is called an FGC-ring. In this paper, we are able to settle one of their conjectures [SWB, p. 1279], thus giving a characterization of reduced FGC-rings with Noetherian maximal ideal spectrum. We show that, for R a reduced ring with Noetherian maximal ideal spectrum, R is an FGC-ring if and only if R is a finite direct sum of h-local Bezout domains and each localization of R is an almost maximal valuation ring. The "if" part of the statement above was shown in [SW] to be an easy consequence of results due to Eben Matlis [M]. Other known results on FGC-rings are reviewed in [SW]; the reader is also referred to [SW] for definitions of the concepts above. The examples constructed in [W] show that our main result answers the FGC question for a large class of rings where it was unknown.

Of course it would be nice to give a complete characterization of reduced FGC-rings. This would require settling another question in [SW]: "Does every FGC-ring have Noetherian maximal ideal spectrum?" They show the answer is "yes" if the ring has fewer than 2^c prime ideals [SW, Corollary 6.8].

1. The main theorem. The result stated in the introduction will easily follow from the Theorem below:

Theorem. *Let R be an FGC-domain. Then every nonzero prime ideal P is contained in a unique maximal ideal.*

Proof. Suppose R contains two distinct maximal ideals M and N with

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$P \subseteq M \cap N$. We show R is not FGC. By localizing R at $R - (M \cup N)$, we may assume M and N are the only maximal ideals of R. Notice that R does not have FGC unless R is locally a valuation ring. (The FGC condition, with Nakayama's lemma, implies for a local ring that any two-generator ideal must be generated by one of the given generators.) Thus we will assume that R_M and R_N are valuation rings.

These remarks follow easily by localization and globalization:

Remarks. (1) If A is an R-submodule of the quotient field of R, then $A = A_M \cap A_N$. (2) If $t \notin P$ and $u \in P$, then $u/t \in P$. (3) If $u \in P$, then $uR_M \subseteq R$, $uR_N \subseteq R$. (4) If $x \in M - N$, $y \in N - M$, then $xR = xR_M \cap R$, $yR = yR_N \cap R$ and $xyR = xR_M \cap yR_N$.

The construction. Fix $x \in M - N$, $y \in N - M$ and $0 \neq z \in P$. Set $I = xyzR$. Then by the Remarks, $I = I_M \cap I_N$, where $I_M = xzR_M$ and $I_N = yzR_N \subseteq R$. Now define S to be the submodule of $R/I_M \oplus R/I_N$ generated by $(\bar{1}, \bar{x})$ and $(\bar{0}, \bar{y})$. Here and in what follows, we write elements of S as ordered pairs in $R/I_M \oplus R/I_N$; bars imply elements of R are replaced by their images in R/I_M or R/I_N. We also use π_1 and π_2 as the natural projections from R to R/I_M and R to R/I_N.

The module $T = R/I_M \oplus R/I_N$ is borrowed from [SW, Proposition 3.3 and Corollary 1.6]; under the same hypotheses as ours, they show T has no cyclic direct summand whose annihilator is equal to the annihilator of T.

Our next step is to calculate S locally. It is convenient to begin by noting the annihilators in R of some elements of S:

$$\operatorname{ann}(\bar{1}, \bar{x}) = I_M \cap I_N = I, \qquad \operatorname{ann}(\bar{0}, \bar{y}) = zR_N \quad \text{and} \quad \operatorname{ann}(\bar{y}, \bar{0}) = I_M.$$

Thus in R_M, $\operatorname{ann}(\bar{1}, \bar{x}) = I_M$, $\operatorname{ann}(\bar{0}, \bar{y}) = zR_N R_M$; in R_N, $\operatorname{ann}(\bar{1}, \bar{x}) = I_N$, $\operatorname{ann}(\bar{y}, \bar{0}) = I_M R_N = zR_M R_N$. Now $S_M = R_M(\bar{1}, \bar{x}) + R_M(\bar{0}, \bar{y})$ and the sum in R_M is easily observed to be direct, so $S_M \cong R_M/I_M \oplus R_M/zR_N R_M$.

Next, $S_N = R_N(\bar{1}, \bar{x}) + R_N(\bar{0}, \bar{y})$; since $(\bar{y}, \bar{0}) = y(\bar{1}, \bar{x}) - x(\bar{0}, \bar{y})$ and $(\bar{0}, \bar{y}) = y(\bar{1}, \bar{x})/x - (\bar{y}, \bar{0})/x$, we have $S_N = R_N(\bar{1}, \bar{x}) + R_N(\bar{y}, \bar{0})$. Now if $r(\bar{1}, \bar{x}) = s(\bar{y}, \bar{0})$ for $r, s \in R_N$, then $rx \in I_N$, and since x is a unit in R_N, $r \in I_N = \operatorname{ann}(\bar{1}, \bar{x})$ in R_N. Thus the sum is direct and $S_N \cong R_N/I_N \oplus R_N/zR_N R_M$.

Claim. If S is a direct sum of cyclics, then either (1) $S \cong R/I \oplus R/zR_M R_N$ or (2) $S \cong R/I_M \oplus R/I_N$.

Proof. First suppose $S = A \oplus B \oplus C \oplus D \oplus \ldots$, each cyclic. Now $S_M \cong R_M/I_M \oplus R_M/zR_N R_M$ and $S_M \cong A_M \oplus B_M \oplus C_M \oplus D_M \oplus \cdots$. But R_M is a valuation ring; thus we can suppose $\operatorname{ann} A_M \subseteq \operatorname{ann} B_M \subseteq \ldots$, and the

decomposition of S_M must be unique. Hence $A_M \cong R_M/I_M$ and $B_M \cong R_M/zR_NR_M$; also $C_M = D_M = \cdots = 0$. However, $A_{NM} = A_{MN} = R_{MN}/I_MR_N$ and $(A_{MN})_P = R_P/zR_P \neq 0$. Therefore $A_N \neq 0$ and similarly $B_N \neq 0$. Now R_N is also a valuation ring, forcing the decomposition $S_N \cong R_N/I_N \oplus R_N/zR_NR_M$ to be unique. Thus either $A_N \cong R_N/I_N$ and $B_N \cong R_N/zR_NR_M$ or $A_N \cong R_N/zR_NR_M$ and $B_N \cong R_N/I_N$; in any case $C_N = D_N = \cdots = 0$, which implies $C = D = \cdots = 0$.

We have now shown that S is a direct sum of *two* cyclics, $S = A \oplus B$ and if we set $A_M \cong R_M/I_M$, then there are two cases:

Case 1. $A_M \cong R_M/I_M$, $A_N \cong R_N/I_N$, $B_M \cong R_M/zR_MR_N$, $B_N \cong R_N/zR_MR_N$.

Case 2. $A_M \cong R_M/I_M$, $A_N \cong R_N/zR_MR_N$, $B_M \cong R_M/zR_MR_N$, $B_N \cong R_N/I_N$.

In Case 1, (ann $A)R_M = I_M$ and (ann $A)R_N = I_N$; thus ann $A = I$. Also ann $B = zR_MR_N \cap zR_MR_N = zR_MR_N$. Thus in Case 1, we have situation (1) of our claim. In Case 2, (ann $A)R_M = I_M$ and (ann $A)R_N = zR_MR_N$, and so ann $A = I_M$; similarly ann $B = I_N$, thus giving situation (2) and completing the proof of the claim.

The remainder of the proof of the Theorem consists of demolishing the two cases of the claim. Supposing Case 2, put $S = Rv \oplus Rw$, where ann $v = I_M$ and ann $w = I_N$. Choose $a, b, c, d \in R$ with $v = (\pi_1(a), \pi_2(ax + by))$, $w = (\pi_1(c), \pi_2(cx + dy))$, where x and y are as in the definition of S.

Now $xz \in$ ann v implies $xz\pi_2(ax + by) = \bar{0}$, that is, $xz(ax + by) \in I_N = yzR_N$, and so $ax^2 + bxy \in yR_N$. Since x^2 is a unit in R_N, we have $a \in yR_N \cap R = yR$ (Remark 4). Thus we may replace a by ay for some new $a \in R$ in the expression for v. A similar argument shows, in the expression for w, c may be replaced by cx for some new $c \in R$. Now $v = (\pi_1(ay), \pi_2(ayx + by))$ and $w = (\pi_1(cx), \pi_2(cx^2 + dy))$. Note that since v and w generate all of S then $Ray + Rcx + I_M = R$, which implies $a \notin M$ and $c \notin N$.

Next we show $zv = yzaw/cx \neq 0$. The first coordinates are clearly the same; also notice $zv \neq 0$ since $zay \notin I_M$. (If $zay \in I_M = xzR_M$, then $ay \in xR_M$ but ay is a unit in R_M.) Checking the second coordinates, we see $z(ayx + by) \in I_N = xzR_N$ and $yza(cx^2 + dy)/cx = zyax + zy^2ad/cx$. But $cx \notin N$ implies both second coordinates are in I_N. Therefore, $Rv + Rw$ cannot be direct, a contradiction to the assumption that Case 2 held.

For the remainder of the proof of the Theorem, we suppose Case 1. Set $v = (\pi_1(a), \pi_2(ax + by))$, $w = (\pi_1(c), \pi_2(cx + dy))$ where $a, b, c, d \in R$, ann $v = I$, ann $w = zR_NR_M$, and $S = Rv \oplus Rw$.

Since $z \in$ ann w, a simple computation shows $c \in xR_M \cap yR_N = xyR$ (Remark 4). Also $\pi_1(Ra) + \pi_1(Rc) = \pi_1(R)$ implies a is a unit in R. Without

loss of generality, we may assume that $v = (\pi_1(1), \pi_2(x + by))$ and w may be rewritten $(\pi_1(fxy), \pi_2(fx^2y + dy))$ for some $f \in R$. Now $(\pi_1(0), \pi_2(y)) \in S = Rv + Rw$; thus $(\pi_1(0), \pi_2(y)) = hv + kw$ for some $h, k \in R$. Then $0 = h + kfxy + xzr/m$, for some $r \in R$, $m \notin M$, or $h = -kfxy = xzr/m$. Also $y = h(x + by) + k(fx^2y + dy) + yzs/n$, for some $s \in R$, $n \notin N$; that is,

$$y = -kfx^2y - \frac{x^2zr}{m} - bkfxy^2 - \frac{xzyr}{m} + kfx^2y + kdy + \frac{yzs}{n}.$$

By cancelling y on each side of the equation, we see d must be a unit in R. This justifies a further revision: now $v = (\pi_1(1), \pi_2(x + by))$ and $w = (\pi_1(fxy), \pi_2(y + fx^2y))$.

Finally, we complete the proof by showing $Rv \cap Rw \neq (0)$. For this, let

$$s = \frac{xz(x + by)}{y^2(1 + fx^2)(1 - fbxy)}, \qquad r = sfxy + \frac{xz}{y(1 + fx^2)}.$$

Notice that r and s are in R, since $1 - fbxy$ is a unit in R and $y^2(1 + fx^2) \notin P$ (using Remark 1). Checking the first coordinates of rv and sw,

$$r\pi_1(1) = \pi_1(r) = \pi_1(sfxy) + \pi_1(xz/y(1 + fx^2)) = s\pi_1(fxy).$$

(For this, notice that $y(1 + fx^2) \notin M$, so all terms are in I_{M°.) Also, a more complicated, but straightforward, computation shows the second coordinates of rv and sw are the same; in fact $r(x + by) = sy(fx^2 + 1)$. Notice that $sw \neq 0$, since $sy(1 + fx^2) \in I_N = yzR_N$ would imply $s(1 + fx^2) \in zR_N$, or equivalently $x(x + by) \in y^2(1 - fbxy)R_N$, but $x^2 \notin yR_N$.

Corollary. *Let R be a reduced ring with Noetherian maximal ideal spectrum. Then R is an FGC-ring if and only if R is a finite direct sum of h-local Bezout domains and each localization of R is an almost maximal valuation ring.*

Proof. See the proof of Corollary (3.4) in [SW].

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