A FIXED POINT CRITERION FOR LINEAR REDUCTIVITY

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ABSTRACT. Let $G$ be a linear algebraic group over an algebraically closed field. If for all actions of $G$ on smooth schemes, the fixed point scheme is smooth, then $G$ is linearly reductive under either of the additional assumptions: (a) the ground field is characteristic zero, or (b) $G$ is connected, reduced, and solvable.

Let $K$ be an algebraically closed field and $G$ a linear algebraic group over $K$. Say $G$ has the smooth fixed point property if for all actions of $G$ on a smooth scheme, the fixed point scheme is also smooth. Fogarty [1] has shown that if $G$ is linearly reductive, then $G$ has the smooth fixed point property. One can ask the converse question: If $G$ is not linearly reductive, is there an action of $G$ on a smooth scheme with a nonsmooth fixed point scheme? In this note we show how to construct such an action for any $G$ that is a split extension by a unipotent subgroup. This gives an affirmative answer to the question for any class of groups where $G$ being not linearly reductive implies $G$ is a split extension by a unipotent subgroup. In particular this includes all groups of characteristic zero and in arbitrary characteristic connected reduced solvable groups.

Let $G$ be a linear algebraic group over $K$ which is a split extension by the unipotent subgroup $U$. We first show that we may assume that $U$ is the direct sum of copies of the additive group. If $G$ modulo a normal subgroup has an action with a nonsmooth fixed point scheme, then certainly $G$ does. Using this we can replace $G$ by $G$ modulo the commutator subgroup of $U$, and hence we may assume $G$ is commutative. In characteristic zero this already implies $U$ is the direct sum of copies of the additive group. In characteristic $p$ there are truncated Witt groups; however, if we take $G$ modulo $p \cdot U$ we may assume $G$ is the direct sum of additive groups by a theorem of Serre [2].

We now construct an action of $G$ on affine space with a nonreduced fixed point scheme. Using our assumption that $G$ is a split extension, we
pick a section for the exact sequence

\[ 0 \to U \to G \to G/U \to 0. \]

Using this section we write elements of \( G \) as ordered pairs \((u, t)\) with \( u \in U \) and \( t \in G/U \). Let \( \rho: G/U \to \text{Aut}(U) \) give the action of \( G/U \) on \( U \).

The multiplication in \( G \) is given by

\[ (u, t) \cdot (u', t') = (u + \rho(t) \cdot u', t \cdot t'). \]

Let \( X = U \times \text{spec} K[x] = \text{spec} K[y_1 \cdots y_n, x] \). In terms of these coordinates on \( U \), let \( \rho(t) \) be given by the matrix \((\rho_{ij}(t))\). The action of \( G \) on \( X \) is given by

\[ (u, t): y_i \mapsto \sum_j \rho_{ij}(t) y_j + u_i x^2, \quad i = 1, \cdots, n, \]

where \( u = (u_1 u_2 \cdots u_n) \). We now verify that this is an action. Let \((u', t') = (u'_1, u'_2, \cdots, u'_n, t)\) be another element of \( G \). Now \( x \) is fixed so there is nothing to do with \( x \).

\[ (u', t'): y_i \mapsto \sum_j \rho_{ij}(t') y_j + u_i x^2, \]

\[ \sum_j \rho_{ij}(t) \left( \sum_k \rho_{jk}(t') y_k + u_j' x^2 \right) + u_i x^2 \sum_k \rho_{ik}(t \cdot t') y_k + \left( \sum_j \rho_{ij}(t) \cdot u_i' + u_i \right) \cdot x^2. \]

But this is also the result of \((u + \rho(t) \cdot u', t \cdot t') = (u, t) \cdot (u', t')\) acting on \((x, y_1, \cdots, y_n)\).

Now \( X \) is affine \( n + 1 \) space and hence smooth. On the other hand, the fixed point scheme of the action by \( G \) is defined by the ideal \( I \) generated by all the elements of the form \( gr - r \) for \( r \in K[y_1, \cdots, y_n, x] \) and \( g \in G \). By setting \( t \) to be the identity and \( u_1 = 1 \) we see that \( x^2 \in I \). But it is clear that \( x \notin I \) and so the fixed point scheme is not reduced.

**BIBLIOGRAPHY**