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A FIXED POINT CRITERION FOR LINEAR REDUCTIVITY

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ABSTRACT. Let G be a linear algebraic group over an algebraically closed field. If for all actions of G on smooth schemes, the fixed point scheme is smooth, then G is linearly reductive under either of the additional assumptions: (a) the ground field is characteristic zero, or (b) G is connected, reduced, and solvable.

Let K be an algebraically closed field and G a linear algebraic group over K. Say G has the smooth fixed point property if for all actions of G on a smooth scheme, the fixed point scheme is also smooth. Fogarty [1] has shown that if G is linearly reductive, then G has the smooth fixed point property. One can ask the converse question: If G is not linearly reductive, is there an action of G on a smooth scheme with a nonsmooth fixed point scheme? In this note we show how to construct such an action for any G that is a split extension by a unipotent subgroup. This gives an affirmative answer to the question for any class of groups where G being not linearly reductive implies G is a split extension by a unipotent subgroup. In particular this includes all groups of characteristic zero and in arbitrary characteristic connected reduced solvable groups.

Let G be a linear algebraic group over K which is a split extension by the unipotent subgroup U. We first show that we may assume that U is the direct sum of copies of the additive group. If G modulo a normal subgroup has an action with a nonsmooth fixed point scheme, then certainly G does. Using this we can replace G by G modulo the commutator subgroup of U, and hence we may assume G is commutative. In characteristic zero this already implies U is the direct sum of copies of the additive group. In characteristic p there are truncated Witt groups; however, if we take G modulo $p \cdot U$ we may assume G is the direct sum of additive groups by a theorem of Serre [2].

We now construct an action of G on affine space with a nonreduced fixed point scheme. Using our assumption that G is a split extension, we

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pick a section for the exact sequence

$$0 \longrightarrow U \longrightarrow G \longrightarrow G/U \longrightarrow 0.$$

Using this section we write elements of G as ordered pairs (u, t) with $u \in U$ and $t \in G/U$. Let ρ: $G/U \longrightarrow \mathrm{Aut}(U)$ give the action of G/U on U. The multiplication in G is given by

$$(u, t) \cdot (u', t') = (u + \rho(t) \cdot u', t \cdot t').$$

Let $X = U \times \operatorname{spec} K[x] = \operatorname{spec} K[y_1 \cdots y_n, x]$. In terms of these coordinates on U, let $\rho(t)$ be given by the matrix $(\rho_{ij}(t))$. The action of G on X is given by

$$(u, t): \begin{cases} x \mapsto x, \\ y_i \mapsto \Sigma_j \rho_{ij}(t) y_j + u_i x^2, \qquad i = 1, \cdots, n, \end{cases}$$

where $u = (u_1 u_2 \cdots u_n)$. We now verify that this is an action. Let $(u', t') = (u'_1, u'_2, \cdots, u'_n, t)$ be another element of G. Now x is fixed so there is nothing to do with x.

$$(u, t): y_i \mapsto \sum_j \rho_{ij}(t) y_j + u_i x^2,$$

$$\begin{aligned}(u', t'): \sum \rho_{ij}(t) y_j + u_i x^2 \mapsto & \sum_j \rho_{ij}(t) \left(\sum_k \rho_{jk}(t') y_k + u'_j x^2 \right) \\ & + u_i x^2 \sum_k \rho_{ik}(t \cdot t') y_k + \left(\sum_j \rho_{ij}(t) \cdot u'_j + u_i \right) \cdot x^2.\end{aligned}$$

But this is also the result of $(u + \rho(t) \cdot u', t \cdot t') = (u, t) \cdot (u', t')$ acting on $(x, y_1, \cdots, y_n)$.

Now X is affine $n + 1$ space and hence smooth. On the other hand, the fixed point scheme of the action by G is defined by the ideal I generated by all the elements of the form $gr - r$ for $r \in K[y_1, \cdots, y_n, x]$ and $g \in G$. By setting t to be the identity and $u_1 = 1$ we see that $x^2 \in I$. But it is clear that $x \notin I$ and so the fixed point scheme is not reduced.

BIBLIOGRAPHY

1. J. Fogarty, *Fixed point schemes*, Amer. J. Math. 95 (1973).

2. J.-P. Serre, *Groupes algébriques et corps de classes*, Publ. Inst. Math. Univ. Nancago, VII, Actualités Sci. Indust., no. 1264, Hermann, Paris, 1959. MR 21 #1973; erratum, 30, p. 1200.

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