POWERS OF MATRICES
WITH POSITIVE DEFINITE REAL PART

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ABSTRACT. For $n$ by $n$ complex matrices $A$ the following two facts are proven by elementary techniques: 1. If $A^m$ is never normal, $m \in \mathbb{I}^+$, then the equation $xA^mx^* = 0$ has a solution $0 \neq x \in \mathbb{C}^n$, $m \in \mathbb{I}^+$; 2. If $H(A) = (A + A^*)/2$ is positive definite, then $H(A^m)$ is positive definite for all $m \in \mathbb{I}^+$ if and only if $A$ is Hermitian.

Let $M_n(\mathbb{C})$ be the class of $n \times n$ complex matrices and denote by $\Sigma$ the class of Hermitian positive definite elements of $M_n(\mathbb{C})$. Define $H(A) = (A + A^*)/2$ the Hermitian part of $A$, so that $A$ is Hermitian if and only if $H(A) = A$. Let $\Pi = \{A \in M_n(\mathbb{C}) : H(A) \in \Sigma\}$.

The field of values of an element $A \in M_n(\mathbb{C})$ will be denoted by

$$F(A) = \{xAx^* : xx^* = 1, x \in \mathbb{C}^n\}.$$

It is well known that

1. $F(A)$ is convex and compact;
2. the spectrum $\sigma(A) \subseteq F(A)$; and
3. $F(UAU^*) = F(A)$ if $U$ is unitary.

When $A$ is normal $F(A)$ coincides with the convex hull of $\sigma(A)$. If $A_0$ is a principal submatrix of $A$, then

$$F(A_0) \subseteq F(A).$$

If $R$ denotes the right complex half-plane, $\Re(z) > 0$, then

$$A \in \Pi \text{ is equivalent to } F(A) \subseteq R.$$

Since $F(A)$ is convex, the condition

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is equivalent to the existence of a $\theta \in [0, 2\pi)$ such that

$$e^{i\theta} A \in \Pi$$

as well as to the nonexistence of a nonzero $x \in \mathbb{C}^n$ such that

$$xAx^* = 0.$$

We shall prove the following two results concerning integral powers of matrices.

**Theorem 1.** If no integral power of $A \in M_n(\mathbb{C})$ is normal, then the equation $xA^mx^* = 0$ has a solution $0 \neq x \in \mathbb{C}^n$, for some positive integer $m$.

**Theorem 2.** Suppose $A \in \Pi$. Then $A^m \in \Pi$ for all $m \in \mathbb{Z}^+$ if and only if $A \in \Sigma$.

Since $\Sigma$ is clearly closed under the raising of its elements to powers, it will suffice to establish the fact that if $A$ is not Hermitian, then there is an $m$ such that $A^m \notin \Pi$. Since $A \in \Pi$, the condition "$A \in \Sigma$" may also be stated: "$A$ is Hermitian".

The proof of Theorem 2 is our primary goal, but, as a step in the proof, we shall establish Theorem 1, a fact of independent interest. Theorem 1 equivalently gives a sufficient condition which implies that $0 \notin F(A^m)$ for some $m$. The requirement that $m$ be positive in Theorems 1 and 2 is of no consequence. An easy computation shows that both conditions $A \in \Pi$ and $0 \in F(A)$ are inherited under inversion. The solvability of $x^*A^mx = 0$ is also considered in [2].

We shall require the aid of a few lemmas. By a straightforward calculation based upon [1] we obtain

**Lemma 1.** If $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$, $c \neq 0$, then $F(A)$ is (i) an ellipse with foci at $a$ and $b$ and eccentricity $e(A) = d/(1 + d^2)^{1/2}$ where $d = |(a - b)/c|$ if $a \neq b$, and (ii) a circle of radius $|c/2|$ about $a$ if $a = b$.

**Lemma 2.** If

$$A = \begin{bmatrix} a & 0 & \cdots & 0 \\ * & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \\ b \end{bmatrix} \in M_k(\mathbb{C})$$

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where $a, b, c \neq 0$ and $b/a$ is not a nonreal root of unity, then there is a nonzero $x \in C^k$ and an $m \in I^+$ such that $xA^mx^* = 0$.

Proof. Since the condition (9) on $A$ is invariant under nonzero scalar multiplication, we assume without loss of generality that $a = 1$ and $b$ is not a nonreal root of unity. An elementary induction on $m$ then shows that

$$A^m = \begin{bmatrix} 1 & 0 & \cdots & 0 & c \sum_{j=0}^{m-1} b^j \\ \ast & 0 & \ast & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ast & \ast & \cdots & \ast \\ 0 & 0 & \cdots & 0 & b^m \end{bmatrix}.$$ 

We shall show that under our assumptions $0 \in F(A^m_0)$ for some $m \in I^+$ where $A^m_0$ is a $2 \times 2$ principal submatrix of $A^m$. By (5) it then follows that $0 \in F(A^m)$ which is equivalent by (7) and (9) to what we wish to show.

If $b = -1$, then $0 \in F(A^m_0)$ because of (2) and (3), and we would be finished. If $b = 1$, then

$$A^m_0 = \begin{bmatrix} 1 & cm \\ 0 & 1 \end{bmatrix},$$

and by Lemma 1, $F(A^m_0)$ is a circle about 1 of radius $|cm/2|$. Since $c \neq 0$, $|cm/2| \to \infty$ as $m \to \infty$, so that for some $m$ the circle $F(A^m_0)$ would be large enough to capture the origin. In this event we would again be done. Thus we may simply assume that $b \neq 0$ is not a root of unity, and we then have

$$A^m_0 = \begin{bmatrix} 1 & c \left( \frac{b^m - 1}{b - 1} \right) \\ 0 & b^m \end{bmatrix}.$$ 

Then by Lemma 1, $F(A^m_0)$ is an ellipse with foci at 1 and $b^m$ and eccentricity

$$e(A^m_0) = \left| \frac{b - 1}{c} \right| \left( 1 + \left| \frac{b - 1}{c} \right|^2 \right)^{-\frac{1}{2}} = e(A_0).$$

Since $0 \in F(A^m_0)$ if and only if $0 \in F(A^{-m}_0)$, we may assume without loss of generality that $|b| \geq 1$. We thus distinguish two cases: $|b| > 1$ and $|b| = 1$. 

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Case I. Assume $|b| > 1$. Then $b^m$ grows arbitrarily far from 1 while $0 < e(A_0^m) < 1$ is a fixed positive constant. Thus the ellipse $F(A_0^m)$ grows arbitrarily large. Since its eccentricity is fixed and one of its foci is fixed at 1, $F(A_0^m)$ must include any fixed point (in particular 0) for some $m$. Thus the lemma is verified in this case.

Case II. Assume $|b| = 1$, $b = e^{i\theta}$. Since $b$ is not a root of 1, $\theta$ is not a rational multiple of $\pi$. As is well known [3] it then follows that there is a sequence of positive integers $m_1, m_2, \ldots, m_k \ldots$ such that $\lim_{k \to \infty} e^{i m_k \theta} = -1$. Since $e(A_0^m) < 1$ is constant, it then follows from (2) that $0 \in F(A_0^m)$ for all $k$ greater than some $K$. Thus the verification of the lemma is complete.

We also make the following easily verified observations:

Lemma 3. If

$$A = \begin{bmatrix} a & 0 & \cdots & 0 & c \\ * & 0 \\ \vdots & \vdots \\ 0 & * & 0 \\ b \end{bmatrix}$$

and $a/b$ is a root of unity not equal to 1, then some integral power of $A$ is diagonal.

Lemma 4. If

$$T = \begin{bmatrix} A_1 & * \\ A_2 & \ddots \\ \vdots & \vdots \\ 0 & \ddots & A_k \end{bmatrix} \in M_n(C)$$

is block triangular, $A_1, \ldots, A_k$ square, then

$$T^m = \begin{bmatrix} A_1^m & * \\ A_2^m & \ddots \\ \vdots & \vdots & \ddots \\ 0 & \ddots & \ddots & A_k^m \end{bmatrix}.$$
We now turn to the proofs of the two theorems mentioned.

Proof of Theorem 1. In view of (3) it suffices to assume $0 \notin \sigma(A)$. We choose a unitary $U$ to reduce $A$ to upper triangular form:

$$UAU^* = T = \begin{bmatrix} t_{11} & \cdots & t_{ij} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_{nn} \end{bmatrix}.$$  

We then have $A^m = U^* T^m U$; $0 \in F(A^m)$ if and only if $0 \in F(T^m)$; and $A^m$ is normal if and only if $T^m$ is diagonal.

We will show that either $0 \in F(T^m)$ for some $m \in I^+$ or $T^m$ is diagonal for some $m \in I^+$, which proves Theorem 1. Suppose there are $q_0 < (n^2 - n)/2$ nonzero entries in $T$ above the diagonal. If $q_0 = 0$, we are done. If $q_0 > 0$, we may choose a principal submatrix $T_0$ of $T$ determined by the set of consecutive indices $j, j + 1, \ldots, k, k \geq 1 + j$ of the form

$$T_0 = \begin{bmatrix} t_{jj} & 0 & \cdots & 0 & c \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & t_{kk} \end{bmatrix}.$$  

Now there are two possibilities. If $t_{kk}/t_{jj}$ is not a nonreal root of unity, then by Lemma 2, $0 \in F(T_0^{m_1})$ for some $m_1 \in I^*$. In this event it then follows from Lemma 4 and (5) that $0 \in F(T^{m_1})$, and we would be finished. If, alternatively, $t_{kk}/t_{jj}$ is a root of unity not equal to 1, then $T_1$ is diagonal for some $m_1 \in I^*$ by Lemma 3. Let $q_1$ now count the nonzero entries above the diagonal of $T^{m_1}$. Because of Lemma 4, $q_1 \leq (n(n - 1) - (k - j)(k - j + 1))/2$. If $q_1 = 0$, we are finished. If not, we next argue on $T^{m_1}$ as we just did on $T$. Pick a principal submatrix $T_{0,m_1}$ of $T^{m_1}$ of the same form as $T_0$. Then determine $m_2 \in I^*$ such that either $0 \in F(T^{m_2})$ or $T_{0,m_1}^{m_2}$ is diagonal. Continue this process to produce a sequence $m_1, m_2, \ldots, m_t \in I^*$. Because of Lemma 4 this process must terminate ($t \leq (n^2 - n)/2$) with either $0 \in F(T^{m_t})$, $m = m_1 \cdots m_t$, or $q_t = 0$ (which means $T^m$ is diagonal). In either event $m = m_1 \cdots m_t$ is the needed integer and the proof is complete.

Proof of Theorem 2. We suppose $A$ is not Hermitian, and we show there is an $m \in I^+$ such that $A^m \notin I^*$, which establishes Theorem 2. If $\sigma(A)$ is not real and positive, it follows from De Moivre's theorem and (3) and (6)
that $A^m \notin \Pi$ for some $m$. Thus we assume $\sigma(A)$ is real and positive, and since our problem is invariant under unitary equivalence, we assume without loss of generality that $A$ is in upper triangular form. Since $A$ is not Hermitian (i.e. not diagonal in this case), this means $A$ has a principal submatrix

$$A_0 = \begin{bmatrix} a & 0 & \cdots & 0 & c \\ * & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & b \\ \end{bmatrix}$$

with $c \neq 0$ and $a$ and $b$ positive. It follows inductively that

$$A_m = A_0 = \begin{bmatrix} a^m & 0 & \cdots & 0 \\ * & c \sum_{j=0}^{m-1} a^j b^{m-j-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & b^m \\ \end{bmatrix}.$$

By Lemma 4, $A_m^m$ is a principal submatrix of $A^m$, and since $c \sum_{j=0}^{m-1} a^j b^{m-j-1}$ is never 0, we have that $A^m$ is never diagonal and thus never normal. From Theorem 1 it then follows that for some $m$, $0 \notin F(A^m)$. In view of (7) and (8) this yields that $A^m \notin \Pi$ as was to be shown.

Example. That the qualification on the existence of solutions to $x A^m x^* = 0$ given in Theorem 1 is in general needed is seen in the following example. Let

$$A = \begin{bmatrix} 1 & \epsilon \\ 0 & z \end{bmatrix}, \quad z = e^{2\pi i/3}.$$

Then

$$A^{3k} = I, \quad A^{3k+1} = A, \quad A^{3k+2} = \begin{bmatrix} 1 & \epsilon(1+z) \\ 0 & \overline{z} \end{bmatrix}. $$

Thus if $\epsilon$ is sufficiently small, $0 \notin F(A^m)$ for all $m \in I^+$. Since $A^{3k} = I$, the condition of Theorem 1 is not satisfied.

Remark 1. It is well known that the containment $F(AB) \subseteq F(A)F(B)$ is not in general valid. Not even the containment $F(A^m) \subseteq F(A)^m$ is in general valid. In fact, according to Theorem 1 the latter containment is very generally invalid in case $0 \notin F(A)$. In this event $0 \notin F(A)^m$ for all $m \in I^+$ while, with the exception of only very special cases, $0 \notin F(A^m)$ for some $m \in I^+$. 
Remark 2. The requirement "for all $m \in I^+$" is necessary in Theorem 2. For any $m \in I^+$, a nonhermitian matrix $A$ may be constructed so that $A^i \in \Pi$ for all $i = 1, \ldots, m$.


REFERENCES


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