AN ASYMPTOTIC FUGLEDE THEOREM

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ABSTRACT. The main result is that if an operator $B$ on Hilbert space "almost" commutes with a normal operator $N$, then $B$ almost commutes with $N^*$ as well. The Theorem is then extended to a Putnam-like version which states that if $B$ almost intertwines two normal operators, then it almost intertwines continuous functions of those operators.

Let $N$ be a normal operator on Hilbert space with spectral measure $E$, and let $B$ be any bounded operator. Two common equivalent formulations of the Fuglede theorem (for example, see [3]) are the following:

(i) If $NB = BN$, then $N^*B = BN^*$.

(ii) If $NB = BN$, then $E(S)B = BE(S)$ for all Borel sets $S$.

Is an approximate version of these statements true? That is, if $\|zVB - BzV\|$ is "small," is $\|N^*B - BN^*\|$ small? Is $\|E(S)B - BE(S)\|$ small? Bastian and Harrison [1] showed that the answer to the second question is "no," by constructing a normal operator $N$, a Borel set $S$, and a sequence $\{B_n\}$ for which $\|B_n\| = 1$ for all $n$ and $\|NB_n - B_nN\| \to 0$, but $\|E(S)B_n - B_nE(S)\| = 1$ for all $n$. Johnson and Williams [4] have shown the existence of a normal operator $N$ and a sequence $\{B_n\}$ such that $\|B_nN - NB_n\| \to 0$ but $\|N^*B_n - B_nN^*\| \geq 1$ for all $n$. Hence the answer to the first question is also negative.

In the Johnson-Williams example, the operators $\{B_n\}$ are not uniformly bounded. The purpose of this note is to show that in case $\|B\|$ is under control, then the answer to the first question is affirmative. Our techniques were inspired by those of Rosenblum [5].

Theorem. Let $N$ be a normal operator. If $K$ and $\epsilon$ are greater than 0, then there is $\delta$ greater than 0 such that $\|B\| \leq K$ and $\|NB - BN\| \leq \delta$ imply $\|N^*B - BN^*\| \leq \epsilon$.

Proof. Without loss of generality we assume that $\|N\| \leq 1$, and that $K = 1$. Let $B$ be an operator with $\|B\| \leq 1$, and set $\|NB - BN\| = \eta$. A
well-known inductive argument shows that

\[ \|N^k B - BN^k\| \leq k\|N\|^{k-1}\|NB - BN\| \leq k\eta, \]

for positive integers \( k \). Thus, for any complex number \( \lambda \),

\[ \|e^{i\lambda N} B - Be^{i\lambda N}\| = \left\| \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} (N^k B - BN^k) \right\| \]

\[ \leq \sum_{k=0}^{\infty} \frac{|\lambda|^k}{k!} \|N^k B - BN^k\| \leq \sum_{k=0}^{\infty} \frac{|\lambda|^k}{k!} k\eta \]

\[ = \eta|\lambda| \sum_{k=1}^{\infty} \frac{|\lambda|^{k-1}}{(k-1)!} = \eta|\lambda|e^{\lambda|\lambda|}. \]

Similarly we can show that \( e^{\lambda N} \) bounds each of the norms \( \|e^{i\lambda N}\| \), \( \|e^{-i\lambda N^*}\| \), and \( \|e^{i\lambda N^*}\| \). Hence

\[ \|e^{i\lambda N^*}(e^{i\lambda N} B - Be^{i\lambda N})e^{-i\lambda N} e^{-i\lambda N^*}\| \leq \eta|\lambda|e^{\lambda|\lambda|}. \]

Since \( N \) is normal we can add exponents to get

\[ \|e^{i(\lambda N + \lambda N^*)} Be^{-i(\lambda N + \lambda N^*)} - e^{i\lambda N^*} Be^{-i\lambda N^*}\| \leq \eta|\lambda|e^{\lambda|\lambda|}. \]

or

\[ \|e^{i\lambda N^*} Be^{-i\lambda N^*}\| \leq \|e^{i(\lambda N + \lambda N^*)} Be^{-i(\lambda N + \lambda N^*)}\| + \eta|\lambda|e^{\lambda|\lambda|}. \]

But \( \lambda N + \lambda N^* \) is Hermitian, so \( e^{i(\lambda N + \lambda N^*)} \) is unitary, and the first term on the right-hand side of the last inequality is equal to \( \|B\| \) for all values of \( \lambda \). Thus

\[ \|e^{i\lambda N^*} Be^{-i\lambda N^*}\| \leq \|B\| + \eta|\lambda|e^{\lambda|\lambda|} \leq 1 + \eta|\lambda|e^{\lambda|\lambda|}. \]

The function \( \phi \) defined by \( \phi(\lambda) = e^{i\lambda N^*} Be^{-i\lambda N^*} \) is an analytic operator-valued function of \( \lambda \). By Cauchy's integral formula we have

\[ \phi'(0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(\lambda)}{\lambda^2} \, d\lambda \]

where \( \Gamma \) is any closed Jordan curve whose interior contains the origin. In case \( \Gamma \) is a circle of radius \( r \) with center at the origin we get the estimate

\[ \|\phi'(0)\| \leq \frac{2\pi r}{2\pi} \cdot \frac{1}{r^2} \max_{\Gamma} \|\phi(\lambda)\| \leq \frac{1}{r} (1 + \eta re^{4r}) = \frac{1}{r} + \eta e^{4r}. \]
Notice that
\[ \phi'(\lambda) = iN^*e^{i\lambda N^*}Be^{-i\lambda N^*} - ie^{i\lambda N^*}BN^*e^{-i\lambda N^*}, \]
so that \( \|\phi'(0)\| = \|N^*B - BN^*\| \). Hence
\[ \|N^*B - BN^*\| \leq 1/r + \eta e^{4r}, \]
and this equation holds for all \( r > 0 \). Thus if \( \epsilon > 0 \), choose \( r = 2/\epsilon \) and choose \( \delta = \frac{1}{2}\eta e^{-4r} \). Then when \( \|NB - BN\| \leq \delta \), that is, \( \eta \leq \delta \), we have
\[ \|N^*B - BN^*\| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon e^{-4r}e^{4r} = \epsilon. \]
This completes the proof.

S. K. Berberian's (by now well-known) trick [2] allows us to extend the result to a Putnam-like version:

**Corollary 1.** Let \( N_1 \) and \( N_2 \) be normal. If \( K \) and \( \epsilon \) are greater than zero, there exists \( \delta \) greater than zero such that \( \|B\| \leq K \) and \( \|N_1B - BN_2\| \leq \delta \) imply \( \|N_1^*B - BN_2^*\| \leq \epsilon. \)

**Proof.** Let \( \mathcal{H} \) be the Hilbert space on which \( N_1 \), \( N_2 \), and \( B \) act. Define operators \( \tilde{N} \) and \( \tilde{B} \) on the space \( \mathcal{H} \oplus \mathcal{H} \) by
\[
\tilde{N} = \begin{pmatrix} N_2 & 0 \\ 0 & N_1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}.
\]

Then \( \|\tilde{B}\| = \|B\| \), \( \|N_1B - BN_2\| = \|\tilde{N}B - \tilde{B}N\| \), and \( \|N_1^*B - BN_2^*\| = \|\tilde{N}^*B - \tilde{B}N\| \). Application of the Theorem to \( \tilde{N} \) and \( \tilde{B} \) gives the corollary.

**Corollary 2.** Let \( N_1 \) and \( N_2 \) be normal and let \( \psi \) be a complex-valued function continuous on the union of the spectra of \( N_1 \) and \( N_2 \). If \( K \) and \( \epsilon \) are greater than zero, then there is \( \delta \) greater than zero such that \( \|B\| \leq K \) and \( \|N_1B - BN_2\| \leq \delta \) imply that \( \|\psi(N_1)B - B\psi(N_2)\| \leq \epsilon. \)

**Proof.** The technique used to prove Corollary 1, plus the observation that \( \psi(N_1 \oplus N_2) = \psi(N_1) \oplus \psi(N_2) \), shows that we may assume that \( N_1 = N_2 = N \); without loss of generality we assume as before that \( \|N\| = 1 \). We use a three-step proof.

**Step one.** In case \( \psi(N) \) is a monomial in \( N \) and \( N^* \), say \( \psi(N) = N^kN^*j \), we have...
\[ \|N^k N^* j B - B N^k N^* j\| \leq \|N^k N^* j B - N^k B N^* j\| + \|N^k B N^* j - B N^k N^* j\| \]
\[ \leq \|N^* j B - B N^* j\| + \|N^k B - B N^k\|, \]

since \( N \) has norm 1. By the relation used at the beginning of the proof of the Theorem,
\[ \|N^k N^* j B - B N^k N^* j\| \leq j \|N^* j - 1\| \|N^* B - B N\| + k \|N\|^{-1} \|N B - B N\| \]
\[ = j \|N^* B - B N^*\| + k \|N B - B N\|. \]

Choose \( \delta' \) so that \( \|B\| < K \) and \( \|N B - B N\| \leq \delta' \) imply \( \|N^* B - B N^*\| \leq \epsilon/2j \), and let \( \delta = \min(\delta', \epsilon/2k) \). Then if \( \|B\| < K \) and \( \|N B - B N\| \leq \delta \) we have
\[ \|N^k N^* j B - B N^k N^* j\| \leq j \cdot \epsilon/2j + k \cdot \epsilon/2k = \epsilon. \]

**Step two.** In case \( \psi(N) \) is a polynomial in \( N \) and \( N^* \), say \( \psi(N) = \sum_{k + j \leq m} \alpha_{k,j} N^k N^* j \), we have
\[ \|\psi(N) B - B \psi(N)\| \leq \sum_{k + j \leq m} \|\alpha_{k,j}\| \|N^k N^* j B - B N^k N^* j\|. \]

By Step one, choose \( \delta_{k,j} \) greater than zero so that \( \|B\| < K \) and \( \|N B - B N\| \leq \delta_{k,j} \) imply \( \|N^k N^* j B - B N^k N^* j\| \leq \epsilon/\|\alpha_{k,j}\| (m + 1) \). Let \( \delta = \min\{\delta_{k,j} : k + j \leq m\} \). Then \( \delta > 0 \) and the conditions \( \|B\| < K \) and \( \|N B - B N\| \leq \delta \) imply
\[ \|\psi(N) B - B \psi(N)\| \leq \sum_{k + j \leq m} \|\alpha_{k,j}\| \cdot \frac{\epsilon}{\|\alpha_{k,j}\| (m + 1)} \]
\[ \leq \sum_{k + j \leq m} \frac{\epsilon}{m(m + 1)} \leq \epsilon. \]

**Step three.** Now let \( \psi \) be any function continuous on the spectrum of \( N \). Then there is a polynomial \( p(\lambda, \bar{\lambda}) \) such that \( \sup_{\lambda \in \Lambda(N)} |p(\lambda, \bar{\lambda}) - \psi(\lambda)| \leq \epsilon/4K \), where \( \Lambda(N) \) is the spectrum of \( N \). We then have
\[ \|\psi(N) B - B \psi(N)\| \leq \|\psi(N) B - p(N, N^*) B\| + \|p(N, N^*) B - B p(N, N^*)\| \]
\[ + \|B p(N, N^*) - B \psi(N)\| \leq 2\|B\| \cdot \|\psi(N) - p(N, N^*)\| \leq \frac{\epsilon}{2}. \]

Finally, by Step two we can choose \( \delta \) greater than zero so that \( \|B\| < K \) and \( \|N B - B N\| \leq \delta \) imply \( \|p(N, N^*) B - B p(N, N^*)\| \leq \epsilon/2 \). The condition defining the polynomial \( p \), plus the spectral theorem, imply that
\[ \| \psi(N) - p(N, N^*) \| \leq \epsilon/4K. \] Hence for \( \|B\| \leq K \) and \( \|NB - BN\| \leq \delta \) we have
\[ \| \psi(N)B - B\psi(N) \| \leq 2K \cdot \epsilon/4K + \epsilon/2 = \epsilon. \]

The proof is complete.

REFERENCES


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