LOCAL EUCLIDEAN FOUR-POINT PROPERTIES WHICH CHARACTERIZE INNER-PRODUCT SPACES

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ABSTRACT. Let $M$ be a complete, convex, externally convex metric space. We show $M$ is an inner-product space if and only if for each point $t$ of $M$, $M$ contains a sphere $S_t$ which has the euclidean queasy, feeble or weak four-point property.

1. Introduction. The purpose of this paper is to give generalizations of the Blumenthal and Day characterizations of inner-product spaces among the class of complete, convex, externally convex metric spaces. For definitions and a detailed study of these concepts, see [3]. A metric space $M$ has the euclidean weak four-point property provided each quadruple $p, q, r, s$ of points of $M$ for which $pq + qr = pr$ is congruently (isometrically) embeddable in the euclidean plane $E_2$. In [3], Blumenthal showed that the euclidean weak four-point property characterizes inner-product spaces among the class of complete, convex, externally convex metric spaces. Blumenthal [4] introduced the euclidean feeble four-point property; namely each quadruple of points $p, q, r, s$ of $M$ for which $pq + qr = pr$ and $pq = qr$ is congruently embeddable in the euclidean plane $E_2$. He then showed that in a complete, convex, externally convex metric space the euclidean feeble four-point property implies the euclidean weak four-point property, thus obtaining another characterization of inner-product spaces. Day [6] showed the weaker condition, the euclidean queasy four-point property; for each pair of distinct points $p, r$ of $M$ there is a point $q, p \neq q \neq r$, between $p$ and $r$ such that for each point $s$ of $M$ the quadruple $p, q, r, s$ is congruently embeddable in $E_2$, characterizes inner-product spaces among the class of complete, externally convex metric spaces. We will say a metric space has the local euclidean weak, feeble, queasy four-point property, respectively, provided for each point $t$ of the space, there is a spherical neighborhood $S_t$ with center $t$, such that

Received by the editors December 4, 1973 and, in revised form, May 22, 1974.
Key words and phrases. Banach space, complete, convex, externally convex, inner-product space, local euclidean four-point properties, metric space.
$S_t$ has the euclidean weak, feeble, or queasy four-point property. The main result of this paper is that a complete, convex, externally convex metric space is an inner-product space if and only if it has one of the aforementioned local four-point properties. It should be noted as in Day's work [6], that if the space has two distinct points, then the local euclidean queasy four-point property already assures the space is locally convex. We should also mention that Busemann [5, p. 51] asserts that every $G$-space which satisfies a local Pasch axiom has the local euclidean weak four-point property. Our main result shows this is false; for the two-dimensional Banach spaces with unique lines are all $G$-spaces which satisfy Pasch's axiom, but only one of them possesses the local euclidean weak four-point property.

Throughout this paper, $M$ will denote a complete, convex, externally convex metric space. Also, for convenience, we will denote the local euclidean queasy four-point property, the local euclidean feeble four-point property, and the local euclidean weak four-point property, respectively by $l.e.q.f.p.p.$, $l.e.f.f.p.p.$ and $l.e.w.f.p.p.$

The following theorem, which is found in Busemann [5, p. 30] is useful in the sequel.

**Theorem B.** If $S(p; \rho/2)$ is a sphere with center $p$ and radius $\rho/2$ and if $x, y$ are elements of $S(p; \rho/2)$ then any segment with endpoints $x$ and $y$ is contained in $S(p; \rho)$.

2. Some immediate consequences of $l.e.q.f.p.p.$ It is easily seen that if $M$ has $l.e.w.f.p.p.$ then $M$ has $l.e.f.f.p.p.$, which in turn implies $M$ has $l.e.q.f.p.p.$ We first show $l.e.q.f.p.p.$ implies that distinct points $a$ and $b$, with distance $ab$ sufficiently small, lie on a unique metric segment. That $l.e.q.f.p.p.$ implies $l.e.w.f.p.p.$ then follows quite easily. Thus we show the three local properties are equivalent.

**Theorem 2.1.** If $t$ is an element of $M$ and if $S(t; \rho)$ has the euclidean queasy four-point property, then each pair of distinct points in $S(t; \rho/2)$ are endpoints of exactly one metric segment.

**Proof.** Since $M$ is complete and convex, distinct points $a, b$ in $S(t; \rho/2)$ are endpoints of at least one segment and by Theorem B, each such segment is contained in $S(t; \rho)$. Suppose $a, b$ are endpoints of at least two segments; say $S_1(a, b)$ and $S_2(a, b)$. Let $c$ be a point of $S_1(a, b) - S_2(a, b)$. In traversing $S_1(a, b)$ from $c$ to $a$, a first point $d$ of $S_2(a, b)$ is encountered, while a first point $e$ of $S_2(a, b)$ is similarly met in traversing $S_1(a, b)$ from $c$ to $b$. Thus the subsegments $S_1(d, e)$ and $S_2(d, e)$ of $S_1(a, b)$ and
$S_2(a, b)$, respectively, have only the endpoints $d$, $e$ in common. The Euclidean queuey four-point property implies the existence of a point $g$ between $d$ and $e$ such that for each point $s$ in $S(t; \rho)$, $d$, $e$, $g$, $s$, is congruent to a quadruple of points of $E_2$. Since $S_1(d, e)$ and $S_2(d, e)$ have only $d$, $e$ in common, $g$ is not on one of the segments; say $S_1(d, e)$. For the particular point $f$ on $S_1(d, e)$ such that $df = dg$ and $ef = eg$, the quadruple $d$, $e$, $f$, $g$ is congruent to a quadruple of $E_2$. This is impossible. Therefore each pair of points of $S(t; \rho/2)$ are endpoints of exactly one segment.

**Theorem 2.2.** If $t$ is an element of $M$ and if $S(t; \rho)$ has the Euclidean queuey four-point property, then $S(t; \rho/2)$ has the Euclidean weak four-point property.

**Proof.** Let $p$, $r$ be points of $S(t; \rho/2)$ and let $Q(p, r)$ be the set of points $x$ between $p$ and $r$ for which the quadruple $p$, $r$, $x$, $s$ is congruently embeddable in $E_2$, for each point $s$ in $S(t; \rho/2)$. It follows from the continuity of the metric that $Q(p, r)$ is closed and hence complete. Moreover, if $x$, $y$ are points of $Q(p, r)$, then for each point $s$ in $S(t; \rho/2)$ each of the quadruples $p$, $r$, $x$, $s$, and $p$, $r$, $y$, $s$, is congruently embeddable in $E_2$. Applying the Euclidean law of cosines to the angle with vertex $p$ and sides $S(p, s)$ and $S(p, r)$, using the points $p, r, s$ and then $p, x, s$ we have

$$\frac{(ps^2 + pr^2 - rs^2)}{2ps \cdot pr} = \frac{(ps^2 + px^2 - sx^2)}{2ps \cdot px}.$$  

Similarly for the quadruple $p$, $r$, $y$, $s$ we have

$$\frac{(ps^2 + pr^2 - rs^2)}{2ps \cdot pr} = \frac{(ps^2 + py^2 - sy^2)}{2ps \cdot py}.$$  

It now follows that $p$, $s$, $x$, $y$ is congruently embeddable in $E_2$. In exactly the same manner it is seen that $r$, $s$, $x$, $y$ is congruently embeddable in $E_2$. Applying the Euclidean queuey four-point property to the points $x$, $y$, we obtain a point $z$ such that for each $s$ in $S(t; \rho/2)$, $x$, $y$, $z$, $s$ is congruently embeddable in $E_2$. Use of the above techniques shows $p$, $r$, $z$, $s$ is congruently embeddable in $E_2$. Thus $Q(p, r)$ is a complete, convex metric space and consequently contains a segment with endpoints $p$ and $r$. By Theorem 2.1 there is only one such segment. It now follows that each quadruple of points of $S(t; \rho/2)$ which contains a linear triple is congruently embeddable in $E_2$; that is $S(t; \rho/2)$ has the Euclidean weak four-point property.

We may now conclude that $M$ has one of the local Euclidean four-point properties if and only if it has all of them.

3. **Uniqueness of lines.** In the preceding section we proved l.e.q.f.p.p. implies unique segments, locally. It is now possible to extend this globally
and show that $M$ has a unique metric line through each pair of its distinct points.

**Theorem 3.1.** If $M$ has l.e.w.f.p.p. and if $a, b$ are distinct points of $M$, then $a, b$ are endpoints of exactly one segment.

**Proof.** If $M$ contains distinct points $a, b$ which are endpoints of two distinct segments, then as in the proof of Theorem 2.1, distinct points $d, e$ and segments $S_1(d, e)$ and $S_2(d, e)$ can be found such that $S_1(d, e)$ and $S_2(d, e)$ have only the endpoints $d, e$ in common. Since $M$ is externally convex the segment $S_1(d, e)$ can be prolonged through $e$ to a point $f$ to obtain a segment $S_1(d, f)$ containing $e$ and such that $df = 2de$. Since $M$ has l.e.w.f.p.p. there is a sphere $S(e; \rho)$ with center $e$ and radius $\rho$ which has the euclidean weak four-point property. Since $e$ is between $d$ and $f$, $e$ is between $f$ and each point that is between $d$ and $e$. Let $p, q$ denote points on $S_1(d, e)$ and $S_2(d, e)$, respectively, such that $pe = qe = \rho/4$ and let $r$ denote the point on $S_1(d, f)$ such that $erf$ holds and $er = \rho/4$. Then the quadruple $p, q, r, e$ is congruently embeddable in $E_2$, contrary to uniqueness of lines in $E_2$. Thus segments are unique.

**Theorem 3.2.** If $M$ has l.e.w.f.p.p., then metric segments admit unique prolongations.

**Proof.** Since $M$ is externally convex, segments admit prolongations. If $M$ contained a segment with endpoints $a, b$ which admitted two prolongations, then $M$ would contain points $a, b, c, d$ with $b$ the midpoint of $a$ and $c$ and $b$ the midpoint of $a$ and $d$. Applying l.e.w.f.p.p., we obtain a sphere $S(b; \rho)$ which has the euclidean weak four-point property. We can now find points $p, q, r$, such that $pb = qb = rb = \rho/2$ and such that $b$ is the midpoint of $p$ and $r$ and also of $q$ and $r$. Now $p, q, r, b$ is congruently embeddable in $E_2$ contradicting the uniqueness of lines in $E_2$.

Combining the results of Theorems 3.1 and 3.2 we have the following result.

**Theorem 3.3.** If $M$ has l.e.w.f.p.p., then each pair of distinct points of $M$ lie on exactly one metric line.

Since the local properties are equivalent, Theorem 3.3 remains valid if l.e.w.f.p.p. is replaced by l.e.f.f.p.p. or l.e.q.f.p.p.

4. The characterization of $M$. Since a local four-point property in a normed linear space is global, it suffices to show $M$ is a Banach space. We show the Banach nature of $M$ by showing l.e.w.f.p.p. implies that for
any three points \( p, q, r \) of \( M \), if \( q', r' \) denote the respective midpoints of \( p \) and \( q \) and \( p \) and \( r \), then \( q' r' = qr/2 \). This will then complete our characterization. For Andalafte and Blumenthal [1] called the latter property the Young postulate and proved a complete metric space with a unique line through each pair of its distinct points is a Banach space if and only if it satisfies the Young postulate. In § 3, we have already shown the uniqueness of lines.

**Theorem 4.1.** Let \( t \) be an element of \( M \) and suppose \( S(t; \rho) \) has the euclidean weak four-point property. If \( p, q, r \) are points in \( S(t; \rho/2) \) and if \( q', r' \) are points on the lines joining \( p \) and \( q \) and \( p \) and \( r \), respectively, such that \( pq = \lambda pq \) and \( pr = \lambda pr \), and if \( q', r' \) lie in \( S(t; \rho/2) \), then \( q' r' = \lambda qr \).

**Proof.** Since \( p, q, q' \) are linear, the quadruple \( p, q', q, r \) is congruently embeddable in \( E_2 \). Thus using the euclidean law of cosines for the cosine of the angle with vertex \( p \) and sides \( S(p, q) \) and \( S(p, r) \), we obtain the same result whether using \( p, q, r \) or \( p, q', r \). Similarly, the quadruple \( p, q, r, r' \) is congruently embeddable in \( E_2 \), so the cosine law gives the same result for the angle at \( p \) when evaluating it with the points \( p, q', r \) or \( p, q', r' \).

Equality of the evaluation by the law of cosines in the first and last triangles yields \( q' r' = \lambda qr \).

**Corollary 4.2.** Suppose \( S(t; \rho) \) has the euclidean weak four-point property and \( p, q, r \) are elements of \( S(t; \rho/2) \). For each \( \lambda, 0 < \lambda < 1 \) and for each point \( s \) between \( q \) and \( r \) if \( q', r' \) are between \( p \) and \( q \) and \( p \) and \( r \), respectively, with \( pq' = \lambda pq \) and \( pr' = \lambda pr \), then there is a point \( s' \) between \( q' \) and \( r' \) that is also between \( p \) and \( s \) and \( ps' = \lambda ps \).

**Proof.** Since \( p, q, r \) lie in \( S(t; \rho/2) \), the segments joining each pair of points lie in \( S(t; \rho) \), by Theorem B. Let \( s' \) be the point on \( S(p, s) \) such that \( ps' = \lambda ps \). Then by Theorem 4.1, \( q' s' = \lambda qs \), \( s' r' = \lambda rs \), and \( q' r' = \lambda qr \). The result now follows.

**Theorem 4.3.** If \( M \) has l.e.w.f.p.p., \( p, x, y \) are points of \( M \), and if \( x', y' \) are the midpoints of \( p \) and \( x \) and \( p \) and \( y \), respectively, then \( x'y' = xy/2 \).

**Proof.** Since \( M \) has l.e.w.f.p.p., a positive number \( \rho \) exists such that \( S(p; \rho) \) has the euclidean weak four-point property. Let \( \Lambda = |\lambda > 0| \) for each \( \mu, 0 < \mu < \lambda \), for all \( s, t \) in \( S(p; \rho/4) \) whenever \( s', t' \) are points on the lines joining \( p \) and \( s \) and \( p \) and \( t \), respectively, such that (i) if \( ps' = \mu \cdot ps \) and \( pt = \mu \cdot pt \), then \( s't' = \mu \cdot st \), and (ii) for each point \( m \) between \( s \) and \( t \),
there is a point \( m' \) between \( s' \) and \( t' \) that satisfies the same betweeness relation as \( p, s, s' \) and \( p u = p \cdot p m \) and \( m' s' = m \cdot m s \). If \( \Lambda \) is not bounded above then the theorem follows. By way of contradiction, suppose \( \Lambda \) is bounded above. By Theorem 4.1 and Corollary 4.2, \( \Lambda \neq \emptyset \). Thus \( \sup \Lambda \) exists, say equals \( l \). It follows from the continuity of the metric that \( l \) is an element of \( \Lambda \).

Let \( q, r \) be any points of \( S(p; \rho/4) \) and let \( u, v \) be points on the rays of \( p \) and \( q \) and \( p \) and \( r \), respectively, such that \( p u = l \cdot p q \) and \( p v = l \cdot p r \).

For each point \( w \) on the segment \( S(u, v) \), there is a sphere \( S(w; \rho_w) \) which has the euclidean weak four-point property. Since \( S(u, v) \) is compact and the collection of spheres \( S(w; \rho_w/2) \), \( w \) on \( S(u, v) \) covers \( S(u, v) \), there is a finite subcollection which covers \( S(u, v) \), say \( S_1(w_1; \rho_1/2), S_2(w_2; \rho_2/2), \ldots, S_n(w_n; \rho_n/2) \). Without loss of generality suppose \( u \neq w \neq v \) \((i = 1, 2, \ldots, n)\) and assume \( u w_i w_{i+1} w_{i+2} \) holds \((i = 1, 2, \ldots, n - 2)\). Note \( u w_i w_2 \) and \( u_{n-1} w_{n-2} \) also hold. Let \( s_0 = u, s_1 = w_1, s_2 \) a point between \( w_1 \) and \( w_2 \) that is in both \( S_1(w_1; \rho_1/2) \) and \( S_2(w_2; \rho_2/2), s_3 = w_2, s_4 \) a point between \( w_2 \) and \( w_3 \) that is in both \( S_2(w_2; \rho_2/2) \) and \( S_3(w_3; \rho_3/2) \), and so forth. We thus obtain a finite collection of points \( \{s_m\} \) of \( S(u, v) \) with three points in the same sphere \( S(w_i; \rho_i) \) for which \( s_i s_i' s_i'' \) subsists and \( p s_i' = \alpha \cdot p s_i \) and \( p s_i = \alpha \cdot p s_i'' \) lie in \( S(w_j; \rho_j) \) whenever \( s_i \) does. Since \( s_i' \) is between \( p \) and \( s_i \), it follows from Corollary 4.2 that the \( s_i' \) \((i = 0, 1, \ldots, m)\) are collinear. Moreover, the points \( p, s_i', s_{i+1}', s_i, s_{i+1} \) are congruently embeddable in \( E_2 \) since the mutual distances determined by them are just a constant multiple of the distances of corresponding points in \( S(p; \rho/2) \). Easy applications of the euclidean weak four-point property using the linear triple \( s_i, s_{i+1}, s_i'' \) and one of \( s_i', s_{i+1}', s_i'' \) and the linear triple \( s_{i+1}, s_i, s_{i+1}' \) and one of \( s_i', s_{i+1}, s_i'' \) shows the above congruent embedding can be extended to see that \( p, s_i', s_{i+1}', s_i, s_{i+1}, s_i'' \) are congruently embeddable in \( E_2 \). Since \( s_i', s_{i+1}, s_i, s_{i+1}' \) lie in the same \( S(w_i; \rho_i/2) \), it follows that \( s_i s_{i+1}' s_{i+2} \) holds \((i = 0, 1, \ldots, m)\). Since \( p u = \alpha \cdot p s_0, \alpha < 1, \) and \( p u = l \cdot p q, \) we have \( p s_0'' = (l/\alpha)pq \) and \( p s_m'' = (l/\alpha)pq \). It is clear from the above construction that if \( l < \mu \) or \( l > \mu \) and \( q, r' \) are points on the segments \( S(p; s_0'') \) and \( S(p; s_m'') \) with \( p q' = \mu \cdot p q, \) \( p r' = \mu \cdot p r \), then \( q' r' = \mu \cdot q r \). This contradicts \( l = \sup \Lambda \). Therefore \( \Lambda \) is not bounded above; the proof is complete.

Applying the result of Andalafte and Blumenthal \([1]\) we now have the following result.
Theorem 4.4. If $M$ has l.e.w.f.p.p., then $M$ is a Banach space.

The following theorem follows from Theorems 2.2 and 4.4.

Theorem 4.5. A complete, convex, externally convex metric space is an inner-product space if and only if it has the local euclidean weak, feeble, or queasy four-point property.

The authors [7] gave a local characterization of inner-product spaces among the class of normed linear spaces. So far as they are aware, the present paper gives the only local characterizations of inner-product spaces among a certain class of metric spaces. The proof given here makes strong use of the Young postulate. It would be nice to have a proof that is independent of that postulate, thus removing the necessity of first showing that local properties imply the metric space is a Banach space. If such a proof were obtained, it would probably be possible to obtain similar local characterizations of hyperbolic and spherical spaces. In [2] Andalafte and Valentine characterized euclidean and hyperbolic spaces among a certain class of metric spaces using intrinsic four-point properties. If such a strong use of the Young postulate could be avoided in the present work, it is conceivable that local intrinsic four-point properties would also effect a characterization of those spaces.

REFERENCES