PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, July 1975

DENSE SUBSETS OF βX

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ABSTRACT. An easy lemma is proved which assures the existence of certain dense subsets of βX on the basis of the existence of similar dense subsets of any compactification of X.

1. **Introduction.** In this paper, we show that, using an easily established property of the Stone extension of a mapping, it is possible to find dense sets in the Stone-Čech compactification βX of a space corresponding to dense sets in any other compactification X^*. Using this device, we obtain several new results, and some greatly simplified proofs of known results.

For example, we show that if X is a separable metric space, then every dense subset of βX is separable. We also show that βX is separable if and only if X has *any* separable compactification, and use this result to obtain a nonseparable space X such that βX is separable. In addition, we obtain conditions ensuring that βX has the Blumberg property.

2. **The basic lemma.** All given spaces are assumed to be completely regular and Hausdorff. If X is any such space, we denote its Stone-Čech compactification, as usual, by βX. If X^* is any compactification of X, $i^*\colon \beta X \to X^*$ will denote the (unique) continuous extension from βX onto X^* of the identity map $i\colon X \to X$.

2.1. **Lemma.** *Let X^* be any compactification of X. Then a set D is dense in βX if and only if $i^*(D)$ is dense in X^*.*

Proof. Suppose $i^*(D)$ is dense in X^*. Then $i^*(\mathrm{cl}_{\beta X} D)$ is a compact subset of X^* containing a dense subset, and is hence all of X^*. It follows that $X \subset \mathrm{cl}_{\beta X} D$ and, hence, $\mathrm{cl}_{\beta X} D = \beta X$; i.e., D is dense in βX.

The converse is immediate.

Received by the editors March 20, 1974.

AMS (MOS) subject classifications (1970). Primary 54D35; Secondary 54A25, 54C30.

Key words and phrases. Dense-separable, Stone-Čech compactification, Blumberg's theorem.

3. **Dense-separable spaces.** A space is called *dense-separable* if every dense subset is separable.

3.1. **Theorem.** *The following conditions on X are equivalent*:

(i) *βX is dense-separable.*

(ii) *Every compactification of X is dense-separable.*

(iii) *X has a dense-separable compactification.*

Proof. (i) implies (ii). Let D be a dense subset of a compactification X^* of X. By Lemma 2.1, $i^{*-1}(D)$ is a dense subset of βX, which, by (i), contains a countable dense subset $\tilde{A}$. Now $i^*(\tilde{A})$ is a countable dense subset of D.

(ii) implies (iii) is obvious.

(iii) implies (i). Let X^* be any dense-separable compactification of X, and suppose D is a dense subset of βX. Then $i^*(D)$ is dense in X^*; let A be a countable dense subset of $i^*(D)$. Let $\tilde{A}$ be a countable subset of D that maps onto A. Then by Lemma 2.1, $\tilde{A}$ is dense in βX, and hence in D.

3.2. **Corollary.** *βX is dense-separable if X is either* (1) *separable metric or* (2) *separable and ordered.*

Proof. (1) A separable metric space has a metric (hence dense-separable) compactification. Hence βX is dense-separable, by Theorem 3.1. (2) If X is a separable ordered space, then X has an ordered (separable) compactification. But a separable ordered space is hereditarily separable (see [6]). Thus, X has a dense-separable compactification, so βX is dense-separable.

The following theorem is proved the same way as Theorem 3.1.

3.3. **Theorem.** *The following conditions on X are equivalent*:

(i) *βX is separable.*

(ii) *Every compactification of X is separable.*

(iii) *X has a separable compactification.*

4. **Blumberg spaces.** A space X is called a *Blumberg space* if for every real valued function f defined on X, there is a dense subset D of X such that f restricted to D is continuous. It was shown in [1] that a metric space is Blumberg if and only if it is a Baire space. Recently, it has been shown [5] that it is consistent with the usual axioms for set theory, including the axiom of choice, that there exist a compact Hausdorff

space that is not Blumberg. The next theorem gives a useful condition on X ensuring that βX is Blumberg.

4.1. **Theorem.** *If X is dense in any Blumberg space, then βX is Blumberg. In particular, βX is Blumberg if X is metrizable.*

Proof. Suppose X is dense in the Blumberg space T, and let X^* be any compactification of T. Clearly, X^* is also Blumberg. Now let f be any real valued function on βX. For each $p \in X^*$, choose a point $x_p \in i^{*-1}(p)$, and define a real valued function g on X^* by setting $g(p) = f(x_p)$. Since X^* is Blumberg, there is a dense subset A of X^*, such that g restricted to A is continuous. Let $\tilde{A} = \{x_p : p \in A\}$. Then by Lemma 2.1, $\tilde{A}$ is dense in βX. Further, f is continuous on $\tilde{A}$, since there it agrees with the continuous function $g \circ i^*$.

Finally, if X is metrizable, X has a metric completion, which is Blumberg [1]. Hence βX is Blumberg. (This last result was also obtained by H. E. White [7], using different methods.)

Remark. In [4] an example is given of a space which can be proved to be Blumberg by Theorem 4.1 but not by any other known method.

4.2. **Lemma.** *Let N be a countably infinite discrete space. A topological space X is a Baire space if and only if, for each $f: X \to N$, there is a dense subset D of X such that f restricted to D is continuous.*

A proof can be found in [4] or [7]. Using this lemma, we can prove the following theorem in the same way as we proved Theorem 4.1.

4.3. **Theorem.** *If the space X has a compactification X^* such that every dense subset of X^* is Baire, then every dense subset of βX is Baire.*

5. **Examples.** We denote the real numbers by R, the rationals by Q, and the integers by N.

5.1. **Example** (see also [2]). By Corollary 3.2, βQ is dense-separable. It follows that $\beta Q \backslash Q$ is dense-separable. Also, since βQ is a dense-separable compactification of $\beta Q \backslash Q$, $\beta(\beta Q \backslash Q)$ is dense-separable.

5.2. **Example.** By Theorem 4.1, βQ is Blumberg. We now show that $\beta Q \backslash Q$ is Blumberg. Let f be any real function on $\beta Q \backslash Q$, and $\{r_i\}$ an enumeration of Q. Define $\tilde{f} : \beta Q \to R$ as the extension of f obtained by setting $\tilde{f}(r_i) = i$ for each i. Since βQ is Blumberg, $\tilde{f}$ is continuous on a dense subset D of βQ. Since $D \cap Q$ is nowhere dense, $D \cap (\beta Q \backslash Q)$ is dense in $\beta Q \backslash Q$. Thus f restricted to this dense subset of $\beta Q \backslash Q$ is continuous.

For the next two examples, see [3] for a different proof.

5.3. **Example.** A separable, compact space that is not dense-separable.

Let I be the closed unit interval, and let $X = I^I$, furnished with the product topology. It is well known that X is separable. Let $Y = \{f \in X: f(x) \neq 0$ for only finitely many values of $x\}$. Now Y is easily seen to be dense in X. But Y is not separable. To see this, let $\{f_i\}$ be any countable subset of Y, and, for each i, set $F_i = \{x \in I: f_i(x) \neq 0\}$. Since F_i is finite for each i, $F = \bigcup_i F_i$ is countable. Choose an $x_0 \in I \backslash F$, and let $U = \{f \in Y: f(x_0) > \frac{1}{2}\}$. Then U is a nonempty open set in Y containing no f_i, so that $\{f_i\}$ is not dense in Y. Thus, X is separable but not dense-separable.

5.4. **Example.** A nonseparable space with separable Stone-Čech compactification.

The space is the space Y of the previous example. Since Y has a separable compactification, namely X, it follows from Theorem 3.3 that βY is separable.

5.5. **Example.** For each infinite cardinal $\gamma \leq 2^{\aleph_0}$, we construct a separable space that is a union of γ pairwise disjoint copies of $\beta N \backslash N$.

Let A be a set of cardinal γ, and let $\{x_\alpha: \alpha \in A\}$ be a dense subset of $R \backslash Q$ such that $x_\alpha \neq x_{\alpha'}$ if $\alpha \neq \alpha'$. For each α, let s_α be a sequence in Q converging to x_α. Then if $\alpha \neq \alpha'$, $s_\alpha \cap s_{\alpha'}$ is finite and each s_α is a countably infinite discrete C^*-embedded subset of Q. Therefore, $X = \bigcup_{\alpha \in A} (\mathrm{cl}_{\beta Q} s_\alpha \backslash Q)$ is a union of γ pairwise disjoint copies of $\beta N \backslash N$. Further, since X is dense in βQ, X is separable (Example 5.1). In fact, since X has a dense-separable compactification, namely βQ, βX is dense-separable.

Question. Is there a separable space which is the union of more than $2^{\aleph_0}$ pairwise disjoint copies of $\beta N \backslash N$?

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